Exam Stochastic processes - WISB362 - Spring 2018

Utrecht University - Monday 16 July 2018, 13:30h - 16:30h

Total number of points: 100.
***NO** books or handwritten material are allowed.*

***Exercise* 1**

A DNA nucleotide has any of 4 values. A standard model for a mutational change of the nucleotide at a specific location is a Markov chain model that supposes that in going from period to period the nucleotide does not change with probability $1-3\alpha$, and if it does change then it is equally likely to change to any of the other 3 values, for some $0 < \alpha < \frac{1}{3}$.

(a) **[4 points]** Write down the transition probability matrix P for this process.

(b) **[2 points]** Let $P_{i,j}$ denote the (i,j)th entry of the matrix P. Argue that $P^n_{i,j} = \frac{1}{3}(1-P^n_{i,i})$.

(c) **[8 points]** Show by induction that

$$P^n_{i,j} = \begin{cases} \frac{1}{4} + \frac{3}{4}(1-4\alpha)^n & \text{if } j = i, \\ \frac{1}{4} - \frac{1}{4}(1-4\alpha)^n & \text{if } j \neq i. \end{cases}$$

(d) **[5 points]** What is the long run proportion of time the chain is in each state?

(e) **[5 points]** What is the expected number of steps the particle takes to return to the starting position?

***Exercise* 2**

Recall that the moment generating function $\phi_X(t)$ of the random variable X is defined for all values t by $\phi_X(t) = \mathbb{E}[e^{tX}]$. Let X_i, $i = 1, \ldots, n$, be independent normal random variables with respective means μ_i and variances σ_i^2. Hence the corresponding moment generating functions equal $\phi_{X_i}(s) = \exp\big(\mu_i s + \frac{1}{2}\sigma_i^2 s^2\big)$. Consider its mean

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i.$$

(a) **[5 points]** Prove, using the moment generating function, that also $\bar{X}$ is normally distributed.

(b) **[5 points]** Determine the mean and variance of $\bar{X}$.

***Exercise* 3**

After various robberies, the owner of a jewellery (that manages the shop by himself) has decided to allow only a maximum of two customers in his shop. Customers arrive at a Poisson rate of 4 per hour, and the successive service times are independent exponential random variables with mean equal to 10 minutes.

(a) **[4 points]** Write the system as a birth-and-death process with S the number of customers present.

(b) **[6 points]** Let P_j be the long-run proportion of time that the process is in state $j \in S$. Show that $P_0 = \frac{9}{19}$.

(c) **[5 points]** Determine the average number of customers present in the jewellery.

(d) **[5 points]** What is the proportion of customers that are allowed to enter the shop?

***Exercise* 4**

Let $\{N(t), t \geq 0\}$ be a Poisson process with rate λ (per hour). Let S_n denote the time of the nth event.

(a) **[7 points]** Suppose the Poisson process is independent of the nonnegative random variable T with mean μ and variance σ^2. Show that $\text{Cov}(T, N(T)) = \lambda\sigma^2$.

(b) **[6 points]** Compute $\mathbb{E}[S_4 \,|\, N(1) = 2]$.

(c) **[6 points]** Suppose two events occurred during the first hour. What is the probability that both occurred during the first 20 minutes?

Let $\{M(t), t \geq 0\}$ be another Poisson process, independent from $\{N(t), t \geq 0\}$, with rate μ (per hour). Consider the combined Poisson process $\{N(t) + M(t), t \geq 0\}$ with rate $\lambda + \mu$ (per hour).

(d) **[6 points]** What is the probability the first two events of the combined process are from the N process?

***Exercise* 5**

Consider a particle that moves on a circle, with states numbered $1, 2, \ldots, n$ (see Figure 1 for the case $n = 5$). The process takes one step clockwise with probability p, and counter-clockwise with probability q, where $p + q = 1$.

(a) **[4 points]** Let $P_{(n)}$ be the transition probability matrix corresponding to the random walk on the circle with n states. Write down the transition probability matrix $P_{(5)}$ for the case that $p = \frac{2}{3}$.

(b) **[6 points]** Consider the above described random walk. Show that this is a time reversible process only for $p = q$.

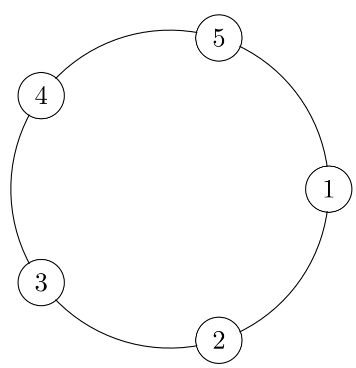

Figure 1: Random walk on a circle with $n = 5$ nodes.

Assume $p \neq q$. Suppose we are interested in the probability that all other positions are visited before the particle returns to its starting state. We can interpret this situation as the gambler's ruin problem, where we are interested in the probability that, starting in state 1, the gambler will reach n before getting broke (getting broke meaning: the particle enters the fictitious state 0 that equals state n).

(c) **[5 points]** Let P_i, $i = 1,...,n$, denote the probability that, starting with i, the gambler's fortune will eventually reach n. Show first that

$$P_{i+1} - P_i = \frac{q}{p}(P_i - P_{i-1}), \qquad (i = 1, 2, \ldots, n-1),$$

and use this result to prove that P_1 satisfies

$$P_1 = \frac{1-(q/p)}{1-(q/p)^n}.$$

(d) **[6 points]** Now we are able to compute the probability that all other positions are visited before the particle returns to its starting state. Starting at a specified state, call it state 1, let T be the time of the first return to state 1. Let A be the event that all states have been visited by time T. Show now that

$$P(A) = \frac{p-q}{1-(q/p)^n} + \frac{q-p}{1-(p/q)^n}.$$

[**Hint**: Condition on the direction of the first step and use the result from (c).]

End of exam

Main probabilistic laws

Discrete laws:

Name	Ω	Probability mass	Mean	Variance
Bernoulli(p)	$\{a,b\}$	$p(b) = p = 1-p(a)$	$a+p(b-a)$	$p(1-p)(a-b)^2$
Binomial(n,p)	$\{0,1,\ldots,n\}$	$p(k) = \binom{n}{k} p^k(1-p)^{n-k}$	np	$np(1-p)$
Geometric(p)	$\mathbb{N}_{\geq 1}$	$p(k) = (1-p)^{k-1}p$	$1/p$	$(1-p)/p^2$
Poisson(λ)	$\mathbb{N}$	$p(k) = \mathrm{e}^{-\lambda}\lambda^k/k!$	λ	λ

Continuous laws:

Name	Probability density	Mean	Variance
Unifom(a,b)	$f(x) = \begin{cases} 1/(b-a) & \text{if } a < x < b \\ 0 & \text{otherwise} \end{cases}$	$(a+b)/2$	$(b-a)^2/12$
Exponential(λ)	$f(x) = \begin{cases} \lambda \mathrm{e}^{-\lambda x} & \text{if } x > 0 \\ 0 & \text{otherwise} \end{cases}$	$1/\lambda$	$1/\lambda^2$
Normal(μ,σ^2)	$f(x) = (1/(\sqrt{2\pi}\,\sigma)\exp\{-(x-\mu)^2/(2\sigma^2)\}$	μ	σ^2