Geometry and Topology – Final exam

- **Write your name and student number clearly on on this exam.**
- You can give solutions in English or Dutch.
- You are expected to explain your answers. (In particular, if you claim that two groups are isomorphic or non-isomorphic, there should be an argument.)
- You are allowed to use results of the lectures, the exercises and homework (and you are also allowed to use the results of part (a) and (b) of a problem in part (c) even if you did not solve (a) and (b)). Furthermore you are allowed to use the results listed below.
- All maps in the statements of the problems are meant to be continuous.
- Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

The following is a list of some of the results mentioned in class or exercises (but maybe not completely proven) that you are allowed to use.

- Let $x \in X$ be a point in a cell complex X and let U be an open neighborhood of x in X. Then there exists a contractible open subset of U containing x.
- Let X and Y be spaces with base points x and y. Assume that x has an open neighborhood U in X such that U deformation retracts onto x and similarly for y. Then $H_n(X \vee Y) \cong H_n(X) \oplus H_n(Y)$ for $n > 0$.
- Every surface has a triangulation.
- For any Δ-complex, singular and simplicial homology agree.

Problem 1 (9 points)**.** *Recall that a subspace* $A \subset X$ *is a* retract of X *if there is a map* $r\colon X \to A$ *with* $r(a) = a$ *for all* $a \in A$*.*

(a) *Show that if* X *is contractible (i.e.* $X \simeq \mathrm{pt}$*), then every retract* $A \subset X$ *of* X *is contractible as well.*

(b) *Give an example of a contractible space* X *and a non-contractible subspace* $A \subset X$*.*

Problem 2 (11 points). *(a) Give the definition of a covering map.*

(b) Give an example of a surjective covering map $X \to Y$ *with* Y *compact and* X *non-compact.*

(c) Is there a surjective covering map $X \to Y$ *with* X *compact and* Y *non-compact?*

Problem 3 (15 points)**.** *Let* $f\colon X \to Y$ *and* $p\colon Z \to Y$ *be maps. Define* f^*Z *to be* $\{(x,z) \in X \times Z : f(x) = p(z)\}$ *with the subspace topology and let* $f^*p\colon f^*Z \to X$ *be the restriction of the projection* $\mathrm{pr}_1\colon X \times Z \to X$.

(a) *Assume that* p *is a covering map. Show that* f^*p *is a covering map as well.*

(b) *Assume that* Y *is a cell complex and let* $f\colon Y^2 \to Y$ *be the inclusion of its* 2*-skeleton (thus we obtain* Y *from* Y^2 *by glueing disks of dimension at least* 3*). Show that for every covering map* $q\colon W \to Y^2$*, there exists a covering* $p\colon Z \to Y$ *and a homeomorphism* $g\colon W \to f^*Z$ *such that* $q = (f^*p) \circ g$.

Continuation of Problem 3

Problem 4 (16 points)**.** *For us, a surface is a compact connected* 2*-dimensional manifold without boundary.*

(a) Give an example of a surface S *with Euler characteristic* -1*.*

(b) Compute the singular homology groups of S*.*

Problem 5 (15 points). *Consider $X = S^1 \vee S^1 \vee S^2$ (where you can choose base points as you prefer).*

(a) *Compute $H_i(X)$ for all $i \geq 0$.*

(b) *Compute $\pi_1(X, x)$ for some base point x of your choice.*

(c) *Find another space Y such that $H_i(X) \cong H_i(Y)$ for all $i \geq 0$, but X is not homotopy equivalent to Y.*

Continuation of Problem 5

Problem 6 (16 points)**.** *Let* $n \geq 1$.

(a) *Let* $f\colon S^n \to S^n$ *be a continuous map. Show that there is an integer* d *such that* $f_*\colon H_n(S^n) \to H_n(S^n)$ *is multiplication by* d. *Call this* d *the* degree *of* f.

(b) *Show that a map* $f\colon S^n \to S^n$ *is surjective if its degree is nonzero.*

(c) *Give for every* $n \geq 1$ *an example of a surjective map* $S^n \to S^n$ *of degree zero.*

Continuation of Problem 6

Problem 7 (18 points)**.** *(a) Let* $n \geq 1$ *and let* $f\colon S^n \to S^n$ *be a map without a fixed point (i.e. without a point* $x \in S^n$ *such that* $f(x) = x$*). Show that the map*

$$S^n \to S^n, \quad x \mapsto -f(x)$$

has degree 1 *in the terminology of Problem 6a.*

(b) Let K *be the Klein bottle and let* $f\colon S^2 \to K$ *and* $g\colon K \to S^2$ *be two maps. Show that the composite* $gf\colon S^2 \to S^2$ *has a fixed point.*

Additional page for continuations of problems