Geometry and Topology – Retake

- **Read the first page carefully.**
- **Write your name and student number clearly on this exam.**
- You can give solutions in English or Dutch.
- You are expected to explain your answers and prove your statements. (In particular, if you claim that two groups are isomorphic or non-isomorphic, there should be an argument.)
- You are allowed to use results of the lectures, the exercises and homework (and you are also allowed to use the results of part (a) of a problem in part (b) even if you did not solve (a)). Furthermore you are allowed to use the results listed below.
- All maps in the statements of the problems are meant to be continuous.
- Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

The following is a list of some of the results mentioned in class or exercises (but maybe not completely proven) that you are allowed to use.

- Let $x \in X$ be a point in a cell complex X and let U be an open neighborhood of x in X. Then there exists a contractible open subset of U containing x.
- Let X and Y be spaces with base points x and y. Assume that x has an open neighborhood U in X such that U deformation retracts onto x and similarly for y. Then $H_n(X \vee Y) \cong H_n(X) \oplus H_n(Y)$ for $n > 0$.
- Every surface has a triangulation.
- For any Δ-complex, singular and simplicial homology agree.
- If S and S' are two orientable or two non-orientable surfaces (compact, without boundary) and their Euler characteristics agree, then S is homeomorphic to S'.

Problem 1 (20 points). *Show that a homotopy equivalence $f\colon X \to Y$ induces a bijection between the set of path components of X and the set of path components of Y and that f restricts to a homotopy equivalence from each path component of X to the corresponding path component of Y.*

Problem 2 (20 points). *Show that there exists a loop $\gamma\colon I \to \mathbb{C}$ with the following properties:*

- $\gamma(0) = \gamma(1) = 1$,
- $\gamma(s) \neq 0$ *and* $\gamma(s) \neq -1$ *for all* $s \in I$,
- *there exists a homotopy* $H\colon I \times I \to \mathbb{C}$ *of loops (i.e.* $H(0,t) = H(1,t) = 1$ *for all* $t \in I$*) from* γ *to* const_1 *such that* $H(s,t) \neq 0$ *for all* $s, t \in I$,
- *but there exists no homotopy* $H\colon I \times I \to \mathbb{C}$ *of loops from* γ *to* const_1 *such that* $H(s,t) \neq 0$ *and* $H(s,t) \neq -1$ *for all* $s, t \in I$.

Problem 3 (20 points)**.** *Let X be the space obtained from S^n by identifying two antipodal points (e.g. $(1, 0, \ldots, 0)$ and $(-1, 0 \ldots, 0)$). Compute the Euler characteristic of X.*

Problem 4 (20 points)**.** *Give a list of spaces such that every path-connected non-empty space X with a covering map $X \to \mathbb{RP}^2 \times \mathbb{RP}^2$ is homeomorphic to exactly one space on the list.*

Problem 5 (36 points). *Let M and N be manifolds of the same dimension $n \geq 2$ and let $\varphi\colon D^n \to M$ and $\psi\colon D^n \to N$ be embeddings. Let $S^{n-1}_r = \{x \in D^n : |x| = r\}$ and $\mathring{D}^n_r = \{x \in D^n : |x| < r\}$. We define the connected sum $M \# N$ as*

$$(M \setminus \varphi(\mathring{D}^n_{1/2})) \sqcup (N \setminus \psi(\mathring{D}^n_{1/2}))/\sim,$$

where $\varphi(x) \sim \psi(x)$ if $x \in S^{n-1}_{1/2}$.

(a) *Let x be a point in M. Show that $H_i(M \setminus \{x\}) \cong H_i(M)$ for $0 < i < n-1$.*

(b) *Show that $H_i(M \# N) \cong H_i(M) \oplus H_i(N)$ for $0 < i < n-1$.*

(c) *Give an example where $H_i(M \# N)$ is not isomorphic to $H_i(M) \oplus H_i(N)$ for some $i > 0$.*

Continuation of Problem 5

Problem 6 (34 points)**.** *Let T be $S^1 \times S^1$ and K be the Klein bottle. Decide for each pair of the following spaces whether they are homotopy equivalent:*

- $\mathbb{RP}^2 \# T$,
- $\mathbb{RP}^2 \# K$,
- T,
- $T \vee S^2$.

Additional page for continuations of problems