Problem 1 (10 points). Call a covering space \( p: X \to Y \) \( n \)-sheeted if \( p^{-1}(y) \) has exactly \( n \) elements for every \( y \in Y \).

(a) Give an example of a 3-sheeted covering space \( X \to Y \) such that \( Y \) is connected, but \( X \) is not.

(b) Let \( T \) be the 2-dimensional torus and \( K \) be the Klein bottle. Construct a 2-sheeted covering \( p: T \to K \).
Problem 2 (10 points). A deformation retract in the weak sense of a space $X$ onto a subspace $A \subset X$ is a homotopy $F : I \times X \rightarrow X$ such that

- $F(0, x) = x$, $\forall x \in X$,
- $F(t, x) \in A$, $\forall x \in A, t \in I$,
- $F(1, x) \in A$, $\forall x \in X$.

If we additionally demand that $F(t, x) = x$ for all $x \in A$ and $t \in I$, we say that $F$ is a deformation retract.

(a) Show that if $X$ deformation retracts onto $A \subset X$ in the weak sense, the inclusion map $\iota : A \rightarrow X$ is a homotopy equivalence.

(b) Give an example of a deformation retract in the weak sense that is not a deformation retract.
**Problem 3** (15 points). Consider the two subsets

$$A_1 = \{(x, y, z) \in \mathbb{R}^3 : z = 0, -1 \leq x, y \leq 1\}$$

and

$$A_2 = \{(x, y, z) \in \mathbb{R}^3 : x = 0, -1 \leq y \leq 1, 0 \leq z \leq 1\}.$$

Let $X \subset \mathbb{R}^3$ be their union (i.e. three rectangles glued along a common edge). Let $x_0 = (0, 0, 0)$, $x_1 = (0, 0, \frac{1}{2})$ and $x_2 = (0, 1, 0)$.

(a) Show that $\pi_1(X \setminus \{x_1\}, x_0) \cong \mathbb{Z}$.

(b) Show that $\pi_1(X \setminus \{x_0\}, x_2) \cong \mathbb{Z} \ast \mathbb{Z}$.

(c) Show that there is no homeomorphism $f : X \to X$ with $f(x_0) = x_1$. 
Continuation of Problem 3
Problem 4 (15 points). Let $S_2$ be a connected sum of the 2-dimensional torus $T$ with itself. Consider an embedding $\phi: D^2 \hookrightarrow T$ and let $T' = T \setminus \phi(\text{int}(D^2))$ (where $\text{int}(D^2)$ is the interior of $D^2$); you are allowed to make the choice of $\phi$ most suitable for you. Consider furthermore

$$E = \left\{ (x,y) \in D^2 : |(x,y - \frac{1}{2})| \geq \frac{1}{4}, |(x,y + \frac{1}{2})| \geq \frac{1}{4} \right\},$$

where $|(x,y)| = \sqrt{x^2 + y^2}$.

Decide for each pair of $S_2, T'$ and $E$ whether they are homeomorphic and whether they are homotopy equivalent.
Additional page for continuations of problems