Geometry and Topology – Midterm

- **Write your name and student number clearly on on this exam.**
- You can give solutions in English or Dutch.
- You are expected to explain your answers.
- You are allowed to use results of the lectures, the exercises and homework (and you are also allowed to use the results of part (a) and (b) of a problem in part (c) even if you did not solve (a) and (b)).
- Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Problem 1 (10 points)**.** *Call a covering space* $p\colon X \to Y$ n-sheeted *if* $p^{-1}(y)$ *has exactly* n *elements for every* $y \in Y$*.*

(a) Give an example of a 3*-sheeted covering space* $X \to Y$ *such that* Y *is connected, but* X *is not.*

(b) Let T *be the* 2*-dimensional torus and* K *be the Klein bottle. Construct a* 2*-sheeted covering* $p\colon T \to K$*.*

Problem 2 (10 points)**.** *A deformation retract in the weak sense of a space X onto a subspace $A \subset X$ is a homotopy $F\colon I \times X \to X$ such that*

- $F(0, x) = x,\ \forall x \in X,$
- $F(t, x) \in A,\ \forall x \in A,\ t \in I,$
- $F(1, x) \in A,\ \forall x \in X.$

If we additionally demand that $F(t, x) = x$ for all $x \in A$ and $t \in I$, we say that F is a deformation retract*.*

(a) Show that if X deformation retracts onto $A \subset X$ in the weak sense, the inclusion map $\iota\colon A \to X$ is a homotopy equivalence.

(b) Give an example of a deformation retract in the weak sense that is not a deformation retract.

Problem 3 (15 points). *Consider the two subsets*

$$A_1 = \{(x, y, z) \in \mathbb{R}^3 : z = 0,\ -1 \leq x, y \leq 1\}$$

and

$$A_2 = \{(x, y, z) \in \mathbb{R}^3 : x = 0, -1 \leq y \leq 1, 0 \leq z \leq 1\}.$$

Let $X \subset \mathbb{R}^3$ *be their union (i.e. three rectangles glued along a common edge). Let* $x_0 = (0, 0, 0)$, $x_1 = (0, 0, \frac{1}{2})$ *and* $x_2 = (0, 1, 0)$.

(a) *Show that* $\pi_1(X \setminus \{x_1\}, x_0) \cong \mathbb{Z}$.

(b) *Show that* $\pi_1(X \setminus \{x_0\}, x_2) \cong \mathbb{Z} * \mathbb{Z}$.

(c) *Show that there is no homeomorphism* $f \colon X \to X$ *with* $f(x_0) = x_1$.

Continuation of Problem 3

Problem 4 (15 points). *Let S_2 be a connected sum of the 2-dimensional torus T with itself. Consider an embedding $\phi\colon D^2 \hookrightarrow T$ and let $T' = T \setminus \phi(\mathrm{int}(D^2))$ (where $\mathrm{int}(D^2)$ is the interior of D^2); you are allowed to make the choice of ϕ most suitable for you. Consider furthermore*

$$E = \left\{(x,y) \in D^2 : |(x, y - \frac{1}{2})| \geq \frac{1}{4}, |(x, y + \frac{1}{2})| \geq \frac{1}{4}\right\},$$

where $|(x,y)| = \sqrt{x^2 + y^2}$.

Decide for each pair of S_2, T' and E whether they are homeomorphic and whether they are homotopy equivalent.

Additional page for continuations of problems