Solution 1

5 pt  (a) Let $A \in U$ and let $H \in \mathbb{R}^n$ be such that $A + H \in U$. Then

$$f(A + H) = f(A) + Df(A)(H) + R(H),$$

with $\|H\|^{-1}R(H) \to 0$ for $H \to 0$. It follows that

$$g(A + H) - g(A) = f(A + H)(A + H) - f(A)A$$

$$= [f(A) + Df(A)(H) + R(H)](A + H) - f(A)A$$

$$= L(H) + r(H),$$

where $L(H) = [Df(A)(H)A + f(A)H]$ is linear in $H$ and where


We note that

$$\|H\|^{-1}\|r(H)\| \leq \|Df(A)\|\|H\| + \|H\|^{-1}\|R(H)\|\|A\| + \|R(H)\| \to 0$$

for $H \to 0$. It follows that $g$ is differentiable at $A$ with total derivative $Dg(A) = L$.

2 pt  (b) By Cramer’s rule, each entry of $A^{-1}$ is of the form $(\det A)^{-1}$ times a matrix $q(A)$ whose entries are polynomial in the entries of $A$. Since clearly, $A \mapsto (\det A)^{-1}$ is $C^1$ on $V := \text{GL}(n, \mathbb{R})$ it follows that $F : A \mapsto A^{-1}$ is $C^1$ on the open subset $V$ of $\text{Mat}(n, \mathbb{R})$. It follows that $F : \text{GL}(n, \mathbb{R}) \to \text{GL}(n, \mathbb{R})$ is differentiable everywhere.

3 pt  (c) It follows from (b) and (a) that $G : X \mapsto F(X)X$ is differentiable with the given formula for the derivative. Since $G(X) = I$ (constant) for every $X$, it follows that $DG(A) = 0$ for all $A \in \text{GL}(n, \mathbb{R})$. Hence, for every $H \in \text{Mat}(n, \mathbb{R})$ we have

$$0 = DG(A)(H) = DF(A)(H)A + F(A)H = DF(A)(H)A + A^{-1}H,$$

so that $DF(A)(H) = -A^{-1}HA^{-1}$ for all $H \in \text{Mat}(n, \mathbb{R})$.

Solution 2

2 pt  (a) By one of the characterisations of submanifold, there exists an open neighborhood $U \ni x^0$ in $\mathbb{R}^n$ and a diffeomorphism from $U$ onto an open subset of $\mathbb{R}^p$ such that $\Phi(M \cap U) = (\mathbb{R}^d \times \{0\}) \cap \Phi(U)$. Now $\psi(y^0) = x^0 \in M \cap U$, so by continuity of $\psi$ there exists an open neighborhood $D^0$ of $y^0$ in $D$ such that $\psi(D^0) \subset U \subset M$. It follows that $\Phi \circ \psi(D^0) \subset \mathbb{R}^d \times \{0\}$.
(b) Since \( \psi|_{D^0} \) and \( \Phi \) are injective, so is their composition \( \Phi \circ \psi \) on \( D^0 \). The composition is \( C^1 \) and at every point \( y \in D^0 \) we have \( D(\Phi \circ \psi)(y) = D\Phi(\psi(y)) \circ D\psi(y) \). The latter is a composition of two injective linear maps, hence injective linear of its own right. Since \( \Phi \circ \psi \) maps into \( \mathbb{R}^d \times \{0\} \), it follows that \( D(\Phi \circ \psi)(y) \) is an injective linear map \( \mathbb{R}^d \to \mathbb{R}^d \times \{0\} \). For dimensional reasons, we see that \( D(\Phi \circ \psi)(y) \) is a bijective linear map \( \mathbb{R}^d \to \mathbb{R}^d \times \{0\} \simeq \mathbb{R}^d \). By the global inverse function theorem it now follows that \( \Phi \circ \psi \) is a diffeomorphism from \( D^0 \) onto an open subset of \( \mathbb{R}^d \times \{0\} \).

(c) In the situation of (b) it follows that \( \psi \) maps \( D^0 \) onto the subset \( \Phi^{-1}(\mathcal{O}) \) of \( M \), which is open in \( M \cap U \) hence in \( M \). Since \( y^0 \) was arbitrary, it follows that \( \psi(D) \) is open in \( M \).

(d) Let now \( V \subset D \) be open. Then \( \psi|_V \) satisfies the above hypothesis, so \( \psi(V) \) is open in \( M \). It follows that \( \psi : D \to \psi(D) \) is a continuous bijection, which is open. Therefore, \( \psi : D \to \psi(D) \) is a homeomorphism. We conclude that \( \psi \) is an embedding.

**Solution 3**

(a) Since \( \rho = (1 + r)^{-n-1} \) on \( \partial B(0; r) \), it follows that
\[
\int_{\partial B(0;r)} \rho(x) \, d_{n-1}x = (1 + r)^{-n-1} \int_{\partial B(0;r)} d_{n-1}x
\]
\[
= (1 + r)^{-n-1} r^n \text{vol}_{n-1}(\partial B(0;1))
\]
\[
\leq r^{-2} \text{vol}_{n-1}(\partial B(0;1)).
\]

(b) The function \( (D_jf)g \) is continuous, hence locally Riemann integrable. We note that
\[
|D_jf(x)g(x)| \leq \|Df(x)\| \cdot |g(x)| \leq \rho(x), \quad (x \in \mathbb{R}^n).
\]
Hence, for every compact Jordan measurable set \( K \subset \mathbb{R}^n \) we have
\[
\int_K |D_jf(x)g(x)| \, dx \leq \int_K M \rho(x) \, dx \leq M \int_{\mathbb{R}^n} \rho(x) \, dx < \infty.
\]
As the number on the right-hand side is independent of \( K \), it follows that \( (D_jf)g \) is absolutely Riemann-integrable. By the same argument, with the roles of \( f \) and \( g \) interchanged, it follows that \( fD_jg \) is absolutely Riemann-integrable over \( \mathbb{R}^n \).

(c) For each \( r > 0 \) the set \( B(0; r) \) is a bounded domain with \( C^1 \)-boundary. Since \( f \) and \( g \) are \( C^1 \)-functions defined on an open neighborhood of \( \bar{B}(0; r) \), it follows by application of Gauss’ theorem that
\[
\int_{B(0;r)} D_jf(x)g(x) \, dx + \int_{B(0;r)} f(x)D_jg(x) \, dx = \int_{\partial B(0;r)} f(x)g(x)n(x)_1 \, d_{n-1}x.
\]
Here \( n \) is the outward unit normal to \( \partial B(0; r) \). Since \((D_j f)g\) and \( f D_j g\) are absolutely integrable, the expression on the left-hand side of the above equation tends to
\[
\int_{\mathbb{R}^n} D_j f(x)g(x)\,dx + \int_{\mathbb{R}^n} f(x)D_j g(x)\,dx
\]
for \( r \to \infty \). The absolute value of the integral on the right-hand side may be estimated by
\[
\int_{\partial B(0,r)} |f(x)| \cdot |g(x)| \, d_{n-1}x \leq \int_{\partial B(0,r)} \rho(x) \, d_{n-1}x \leq r^{-2} \text{vol}_{n-1}(B(0; 1)).
\]
The latter expression tends to zero for \( r \to \infty \). Therefore, the required result follows by taking limits.

**Solution 4**

3 pt  
(a) The map \( \Phi \) is \( C^1 \) since \((r,y) \mapsto r \) and \((r,y) \mapsto \psi(y)\) are \( C^1\)-maps. If \( \Phi(r,y) = \Phi(r',y') \) we find, by taking norms that \( r = r' \) and then \( \psi(y) = \psi(y') \). Since \( \psi \) is injective, we see that \( \Phi \) is injective. The derivative of \( \Phi \) is given by
\[
D\Phi(r,y) = (\psi(y) \mid rD\psi(y)).
\]
We note that the determinant of the derivative equals
\[
\det D\Phi(r,y) = r^{n-1} \det(\psi(y) \mid D\psi(y)).
\]
Since \( D\psi(y) \) has image \( T_{\psi(y)} = \psi(y)^\perp \), it follows that \( D\Phi(r,y) \) is surjective, hence bijective, for all \((r,y) \in (0, \infty) \times D\). It follows from the global version of the inverse function theorem that \( \Phi \) is a diffeomorphism from \((0, \infty) \times D\) onto an open subset of \( V \) of \( \mathbb{R}^n \).

3 pt  
(b) By the substitution theorem we have
\[
\int_V f(x)\,dx = \int_{(0,\infty) \times D} f(\Phi(r,y)) |\det D\Phi(r,y)| \, d(r,y)
\]
\[
= \int_{(0,\infty)} \int_D f(r\psi(y)) r^{n-1} \det(\psi(y) \mid D\psi(y)) \, dy \, dr
\]
and (b) follows.

4 pt  
(c) Since \( \psi(y) \) is a unit normal to \( S \), it follows that
\[
|\det(\psi(y) \mid D\psi(y))| \, dy = \psi^* (d_{n-1}z)_y
\]
so that the integral on the right-hand side of (c) becomes
\[
\int_{(0,\infty)} \int_{\psi(D)} f(rz) r^{n-1} d_{n-1}z \, dr = \int_{(0,\infty)} \int_S f(rz) r^{n-1} d_{n-1}z \, dr.
\]
The latter identity follows from the fact that for each \( r \in (0, \infty) \) the function \( z \mapsto f(rz) \) has support in \( \mathbb{V} \cap S = \psi(D) \).
(d) Let \( a \in \text{supp} f \) then \( a \neq 0 \) hence there exists an embedding \( \psi_a : D_a \to S \) with image containing \( a/\|a\| \). By (b) the set \( V_a := \mathbb{R}_{>0}\psi_a(D_a) \) is an open neighborhood of \( a \). By compactness of \( \text{supp} f \) there exists a (finite) partition of unity \( \{ \chi_j \} \) subordinate to the sets \( V_a \). So \( \text{supp} \chi_j \subset V_{a_j} \) for some \( a_j \). The function \( f_j := \chi_j f \) belongs to \( C_c(V_{a_j}) \), hence it follows by (c) that

\[
\int_{\mathbb{R}^n} f_j(x) \, dx = \int_{(0,\infty)} \int_S f_j(rz) r^{n-1} \, d_{n-1} z \, dr.
\]

Summing over \( j \) we obtain the desired identity.