Exam Analyse in Meer Variabelen 2019-06-26, 9:00–12:00

10 pt total

Solution 1

5 pt (a) Let $A \in U$ and let $H \in \mathbb{R}^n$ be such that $A + H \in U$. Then

$$f(A+H) = f(A) + Df(A)(H) + R(H),$$

with $\|H\|^{-1}R(H) \to 0$ for $H \to 0$. It follows that

$$\begin{aligned} g(A+H) - g(A) &= f(A+H)(A+H) - f(A)A \\ &= [f(A) + Df(A)(H) + R(H)](A+H) - f(A)A \\ &= L(H) + r(H), \end{aligned}$$

where $L(H) = [Df(A)(H)A + f(A)H]$ is linear in H and where

$$r(H) = Df(A)(H)H + R(H)(A+H).$$

We note that

$$\|H\|^{-1}\|r(H)\| \leq \|Df(A)\|\|H\| + \|H\|^{-1}\|R(H)\|\|A\| + \|R(H)\| \to 0$$

for $H \to 0$. It follows that g is differentiable at A with total derivative $Dg(A) = L$.

2 pt (b) By Cramer's rule, each entry of A^{-1} is of the form $(\det A)^{-1}$ times a matrix $q(A)$ whose entries are polynomial in the entries of A. Since clearly, $A \mapsto (\det A)^{-1}$ is C^1 on $V := \mathrm{GL}(n,\mathbb{R})$ it follows that $F : A \mapsto A^{-1}$ is C^1 on the open subset V of $\mathrm{Mat}(n,\mathbb{R})$. It follows that $F : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ is differentiable everywhere.

3 pt (c) It follows from (b) and (a) that $G : X \mapsto F(X)X$ is differentiable with the given fomula for the derivative. Since $G(X) = I$ (constant) for every X, it follows that $DG(A) = 0$ for all $A \in \mathrm{GL}(n,\mathbb{R})$. Hence, for every $H \in \mathrm{Mat}(n,\mathbb{R})$ we have

$$0 = DG(A)(H) = DF(A)(H)A + F(A)H = DF(A)(H)A + A^{-1}H,$$

so that $DF(A)(H) = -A^{-1}HA^{-1}$ for all $H \in \mathrm{Mat}(n,\mathbb{R})$.

Solution 2

2 pt (a) By one of the characterisations of submanifold, there exists an open neighborhood $U \ni x^0$ in $\mathbb{R}^n$ and a diffeomorphism from U onto an open subset of $\mathbb{R}^n$ such that $\Phi(M \cap U) = (\mathbb{R}^d \times \{0\}) \cap \Phi(U)$. Now $\psi(y^0) = x^0 \in M \cap U$, so by continuity of ψ there exists an open neigborhood D^0 of y^0 in D such that $\psi(D^0) \subset U \subset M$. It follows that $\Phi \circ \psi(D^0) \subset \mathbb{R}^d \times \{0\}$.

4 pt (b) Since $\psi|_{D^0}$ and Φ are injective, so is their composition $\Phi \circ \psi$ on D^0. The composition is C^1 and at every point $y \in D^0$ we have $D(\Phi \circ \psi)(y) = D\Phi(\psi(y)) \circ D\psi(y)$. The latter is a composition of two injective linear maps, hence injective linear of its own right. Since $\Phi \circ \psi$ maps into $\mathbb{R}^d \times \{0\}$, it follows that $D(\Phi \circ \psi)(y)$ is an injective linear map $\mathbb{R}^d \to \mathbb{R}^d \times \{0\}$. For dimensional reasons, we see that $D(\Phi \circ \psi)(y)$ is a bijective linear map $\mathbb{R}^d \to \mathbb{R}^d \times \{0\} \simeq \mathbb{R}^d$. By the global inverse function theorem it now follows that $\Phi \circ \psi$ is a diffeomorphism from D^0 onto an open subset of $\mathbb{R}^d \times \{0\}$.

2 pt (c) In the situation of (b) it follows that ψ maps D^0 onto the subset $\Phi^{-1}(\mathscr{O})$ of M, which is open in $M \cap U$ hence in M. Since y^0 was arbitary, it follows that $\psi(D)$ is open in M.

2 pt (d) Let now $V \subset D$ be open. Then $\psi|_V$ satisfies the above hypothesis, so $\psi(V)$ is open in M. It follows that $\psi : D \to \psi(D)$ is a continuous bijection, which is open. Therefore, $\psi : D \to \psi(D)$ is a homeomorphism. We conclude that ψ is an embedding.

Solution 3

2 pt (a) Since $\rho = (1+r)^{-n-1}$ on $\partial B(0;r)$, it follows that

$$\begin{aligned}\int_{\partial B(0;r)} \rho(x)\, d_{n-1}x &= (1+r)^{-n-1} \int_{\partial B(0;r)} d_{n-1}x \\ &= (1+r)^{-n-1} r^{n-1} \mathrm{vol}_{n-1}(\partial B(0;1)) \\ &\leq r^{-2} \mathrm{vol}_{n-1}(\partial B(0;1)).\end{aligned}$$

4 pt (b) The function $(D_j f)g$ is continuous, hence locally Riemann integrable. We note that

$$|D_j f(x) g(x)| \leq \|Df(x)\| \cdot |g(x)| \leq \rho(x), \qquad (x \in \mathbb{R}^n).$$

Hence, for every compact Jordan measurable set $K \subset \mathbb{R}^n$ we have

$$\int_K |D_j f(x) g(x)|\, dx \leq \int_K M\rho(x)\, dx \leq M \int_{\mathbb{R}^n} \rho(x)\, dx < \infty.$$

As the number on the right-hand side is independent of K, it follows that $(D_j f)g$ is absolutely Riemann-integrable. By the same argument, with the roles of f and g interchanged, it follows that $f D_j g$ is absolutely Riemann-integrable over $\mathbb{R}^n$.

4 pt (c) For each $r > 0$ the set $B(0;r)$ is a bounded domain with C^1-boundary. Since f and g are C^1-functions defined on an open neighborhood of $\bar{B}(0;r)$, it follows by application of Gauss' theorem that

$$\int_{B(0;r)} D_j f(x) g(x)\, dx + \int_{B(0;r)} f(x) D_j g(x)\, dx = \int_{\partial B(0;r)} f(x) g(x) \mathbf{n}(x)_1\, d_{n-1}x.$$

Here $\mathbf{n}$ is the outward unit normal to $\partial B(0;r)$. Since $(D_j f)g$ and $fD_j g$ are absolutely integrable, the expression on the left-hand side of the above equation tends to

$$\int_{\mathbb{R}^n} D_j f(x)g(x)\,dx + \int_{\mathbb{R}^n} f(x)D_j g(x)\,dx$$

for $r \to \infty$. The absolute value of the integral on the right-hand side may be estimated by

$$\int_{\partial B(0;r)} |f(x)|\cdot|g(x)|\,d_{n-1}x \leq \int_{\partial B(0;r)} \rho(x)\,d_{n-1}x \leq r^{-2}\mathrm{vol}_{n-1}(B(0;1)).$$

The latter expression tends to zero for $r \to \infty$. Therefore, the required result follows by taking limits.

Solution 4

3 pt (a) The map Φ is C^1 since $(r,y) \mapsto r$ and $(r,y) \mapsto \psi(y)$ are C^1-maps. If $\Phi(r,y) = \Phi(r',y')$ we find, by taking norms that $r = r'$ and then $\psi(y) = \psi(y')$. Since ψ is injective, we see that Φ is injective. The derivative of Φ is given by

$$D\Phi(r,y) = (\psi(y) \mid rD\psi(y)).$$

We note that the determinant of the derivative equals

$$\det D\Phi(r,y) = r^{n-1}\det(\psi(y) \mid D\psi(y)).$$

Since $D\psi(y)$ has image $T_{\psi(y)} = \psi(y)^{\perp}$, it follows that $D\Phi(r,y)$ is surjective, hence bijective, for all $(r,y) \in (0,\infty)\times D$. It follows from the global version of the inverse function theorem that Φ is a diffeomorphism from $(0,\infty)\times D$ onto an open subset of V of $\mathbb{R}^n$.

3 pt (b) By the substitution theorem we have

$$\begin{aligned}\int_V f(x)\,dx &= \int_{(0,\infty)\times D} f(\Phi(r,y))\,|\det D\Phi(r,y)|\,d(r,y)\\ &= \int_{(0,\infty)}\int_D f(r\psi(y))\,r^{n-1}\det(\psi(y) \mid D\psi(y))|\,dy\,dr\end{aligned}$$

and (b) follows.

4 pt (c) Since $\psi(y)$ is a unit normal to S, it follows that

$$|\det(\psi(y) \mid D\psi(y))|\,dy = \psi^*(d_{n-1}z)_y$$

so that the integral on the right-hand side of (c) becomes

$$\int_{(0,\infty)}\int_{\psi(D)} f(rz)\,r^{n-1}\,d_{n-1}z\,dr = \int_{(0,\infty)}\int_S f(rz)\,r^{n-1}\,d_{n-1}z\,dr.$$

The latter identity follows from the fact that for each $r \in (0,\infty)$ the function $z \mapsto f(rz)$ has support in $V \cap S = \psi(D)$.

4 pt (d) Let $a \in \operatorname{supp} f$ then $a \neq 0$ hence there exists an embedding $\psi_a : D_a \to S$ with image containing $a/\|a\|$. By (b) the set $V_a := \mathbb{R}_{>0}\psi_a(D_a)$ is an open neighborhood of a. By compactness of $\operatorname{supp} f$ there exists a (finite) partition of unity $\{\chi_j\}$ subordinate to the sets V_a. So $\operatorname{supp}\chi_j \subset V_{a_j}$ for some a_j. The function $f_j := \chi_j f$ belongs to $C_c(V_{a_j})$, hence it follows by (c) that

$$\int_{\mathbb{R}^n} f_j(x)\, dx = \int_{(0,\infty)} \int_S f_j(rz)\, r^{n-1}\, d_{n-1}z\, dr.$$

Summing over j we obtain the desired identity.