Exam Analyse in Meer Variabelen 2019-06-26, 9:00–12:00

- Write your **name** on every sheet, and on the first sheet your **student number** and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Justify your anwers with complete arguments, unless specified otherwise. If you use results from the books or lecture notes, always **refer to them**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 44. The final grade will be obtained from your total score through division by 4, but not higher than 10.
- You are free to write the solutions either in English, or in Dutch.

Good Luck !

10 pt total **Exercise 1.**

5 pt (a) Let $U \subset \mathrm{Mat}(n,\mathbb{R})$ be open and $f : U \to \mathrm{Mat}(n,\mathbb{R})$ differentiable everywhere. Show that the map $g : U \to \mathrm{Mat}(n,\mathbb{R})$ defined by $g(X) = f(X)X - I$ is differentiable everywhere and that for every $A \in U$ the derivative $Dg(A)$ is the linear map $\mathrm{Mat}(n,\mathbb{R}) \to \mathrm{Mat}(n,\mathbb{R})$ given by

$$Dg(A)(H) = [Df(A)(H)]A + f(A)H, \qquad (H \in \mathrm{Mat}(n,\mathbb{R})).$$

In the following you may use that $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathrm{Mat}(n,\mathbb{R})$.

2 pt (b) Prove that the function $F : \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$, $X \mapsto X^{-1}$ is differentiable at every $A \in \mathrm{GL}(n,\mathbb{R})$.

3 pt (c) For $A \in \mathrm{GL}(n,\mathbb{R})$, determine the total derivative $DF(A) : \mathrm{Mat}(n,\mathbb{R}) \to \mathrm{Mat}(n,\mathbb{R})$.

10 pt total **Exercise 2.** Let M be a C^1-submanifold of dimension d in $\mathbb{R}^n$. Let $D \subset \mathbb{R}^d$ be an open subset and $\psi : D \to \mathbb{R}^n$ an injective C^1-immersion such that $\psi(D) \subset M$.

2 pt (a) For $y^0 \in D$ and $x^0 = \psi(y^0)$, show that there exist an open neighborhood D^0 of y^0 in D, an open neighborhood U of x^0 in $\mathbb{R}^n$ and a diffeomorphism Φ from U onto an open subset of $\mathbb{R}^n$ such that $\psi(D^0) \subset U$ and $\Phi \circ \psi(D^0) \subset \mathbb{R}^d \times \{0\}$.

4 pt (b) For y^0, D^0, U and Φ as in (a), show that $\Phi \circ \psi$ is a diffeomorphism from D^0 onto an open neighborhood $\mathscr{O}$ of $\Phi(x^0)$ in $\mathbb{R}^d \times \{0\}$.

2 pt (c) Show that $\psi(D)$ is open in M.

2 pt (d) Show that ψ is an embedding.

10 pt total **Exercise 3.** In this exercise we assume that $n \geq 1$ and that $\rho : \mathbb{R}^n \to \mathbb{R}$ is the function given by $\rho(x) = (1 + \|x\|)^{-n-1}$, for $x \in \mathbb{R}^n$.

2 pt (a) Show that for all $r > 0$ we have

$$\int_{\partial B(0;r)} \rho(x)\, d_{n-1}x \leq r^{-2} \mathrm{vol}_{n-1}(\partial B(0;1)).$$

From the course exercises you know that the function ρ is absolutely integrable over $\mathbb{R}^n$; this may be used in the sequel. Let $f, g : \mathbb{R}^n \to \mathbb{R}$ be C^1-functions such that for all $x \in \mathbb{R}^n$ we have

$$|f(x)| + \|Df(x)\| \leq \rho(x)^{1/2}, \quad |g(x)| + \|Dg(x)\| \leq \rho(x)^{1/2}.$$

4 pt (b) Show that for each $1 \leq j \leq n$ the functions $(D_j f)g$ and $f(D_j g)$ are absolutely Riemann integrable over $\mathbb{R}^n$.

4 pt (c) Show that

$$\int_{\mathbb{R}^n} D_j f(x)\, g(x)\, dx = -\int_{\mathbb{R}^n} f(x)\, D_j g(x)\, dx.$$

14 pt total **Exercise 4.** In this exercise we assume that $n \geq 2$ and that $S := \{x \in \mathbb{R}^n \mid \|x\| = 1\}$ is the unit sphere in $\mathbb{R}^n$. You may use that S is a C^1-submanifold of $\mathbb{R}^n$ of dimension $n - 1$.

We now assume that $D \subset \mathbb{R}^{n-1}$ is open and that $\psi : D \to \mathbb{R}^n$ is an injective C^1-immersion with image contained in S.

3 pt (a) Show that the map $\Phi : (r, y) \mapsto r\psi(y)$ is a C^1-diffeomorphism from $(0, \infty) \times D$ onto an open subset V of $\mathbb{R}^n$. Hint: you may use that $\psi(y) \perp T_{\psi(y)}S$.

3 pt (b) Show that for every function $f \in C_c(V)$ we have

$$\int_V f(x)\, dx = \int_0^\infty \int_D f(r\psi(y))\, r^{n-1} |\det(\psi(y) \mid D\psi(y))|\, dy\, dr.$$

4 pt (c) Show that for every function $f \in C_c(V)$ we have

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_S f(rz)\, r^{n-1} d_{n-1}z\, dr.$$

Hint: you may use that $\psi(y)$ has length 1 and satisfies $\psi(y) \perp T_{\psi(y)}S$.

4 pt (d) Show that for every $f \in C_c(\mathbb{R}^n \setminus \{0\})$ we have

$$\int_{\mathbb{R}^n} f(x)\, dx = \int_0^\infty \int_S f(rz)\, r^{n-1} d_{n-1}z\, dr.$$