Solution 1

(a) The map G is readily seen to be linear. Therefore, it is differentiable at every element $X^0 \in \mathrm{Mat}(n,\mathbb{R})$, with derivative equal to $DG(X^0) : H \mapsto G(H) = YH$.

(b) Since f is differentiable at I, it follows that for $H \in \mathrm{Mat}(n,\mathbb{R})$ we have $f(I+H) = f(I) + Df(I)(H) + R(H)$, with $\|H\|^{-1}R(H) \to 0$ for $H \to 0$.

It follows from this that

$$\begin{aligned} g(I+H) - g(I) &= (I+H)f(I+H) - f(I) \\ &= (I+H)(f(I) + Df(I)(H) + R(H)) - f(I) \\ &= Hf(I) + Df(I)(H) + R_g(H) \end{aligned}$$

with

$$R_g(H) = HDf(I)(H) + (I+H)R(H).$$

Now $L : H \mapsto Hf(I) + Df(I)(H)$ is linear from $\mathrm{Mat}(n,\mathbb{R})$ to itself, and

$$\|H\|^{-1}\|R_g(H)\| \leq \|Df(I)\|\|H\| + (\|I\| + \|H\|)\|H\|^{-1}\|R(H)\| \to 0$$

so that g is differentiable at I with derivative given by $Dg(I) = L$.

(c) Since F is C^1 it follows that the function F is differentiable at I. Clearly, the function $X \mapsto XF(X)$ equals the function G of (a). By (a) we have $DG(I)(H) = HY$ and by (b) we have

$$DG(I)(H) = DF(I)(H) + YH.$$

Combining these, it follows that the derivative $DF(I)$ is the linear map $\mathrm{Mat}(n,\mathbb{R}) \to \mathrm{Mat}(n,\mathbb{R})$ given by

$$DF(I)(H) = HY - YH.$$

Solution 2

(a) By one of the characterizations of submanifold, there exists an open neighborhood $U \ni x^0$ in $\mathbb{R}^n$ and a submersion $f : U \to \mathbb{R}$ such that $U \cap M = f^{-1}(0)$. Since f is submersive, $Df(x^0) : \mathbb{R}^n \to \mathbb{R}$ is surjective. This is equivalent to $Df(x^0) \neq 0$.

(b) We observe that $T_{x^0}M = \ker Df(x^0)$, so $Df(x^0)c'(0) \neq 0$. By the chain rule we now find that

$$(f \circ c)'(0) = Df(x^0)c'(0) \neq 0.$$

(c) By continuity, there exists a constant $r > 0$ such that $|s|, |t| < r \Rightarrow c(t) + sv \in U$. We consider the function $F : (-r,r) \times (-r,r) \to \mathbb{R}$ given by

$$F(s,t) = f(c(t) + sv).$$

Then $c(t)+sv \in M \iff F(s,t)=0$. We note that F is C^1, with partial derivatives

$$\partial_s F(0,0) = Df(x^0)(v), \qquad \partial_t F(0,0) = Df(x^0)c'(0) \neq 0.$$

By application of the implicit function theorem there exist $0 < \delta, \varepsilon < r$ such that for all $s \in (-\delta, \delta)$ there is a unique $t = t(s) \in (-\varepsilon, \varepsilon)$ such that $F(s,t) = 0$ which is equivalent to $c(t)+sv \in M$.

(d) By the implicit function theorem, δ and ε can be chosen such that $s \mapsto t(s)$ is a C^1-function on $(-\delta, \delta)$. By the implicit function theorem it follows that

$$t'(0) = -(\partial_t F(0,0))^{-1}\partial_s F(0,0) = -[Df(x^0)c'(0)]^{-1}Df(x^0)v$$

From this equality if follows that

$$t'(0)Df(x^0)c'(0) = -Df(x^0)v$$

which is equivalent to $Df(x^0)[t'(0)c'(0)+v] = 0$ hence to $t'(0)c'(0)+v \in T_{x^0}(M)$.

Alternatively, applying implicit differentiation one finds

$$0 = \partial_s|_{s=0} F(t(s),s)| = \partial_s|_{s=0} f(c(t(s)+sv) = Df(x^0)(c'(0)t'(0)+v)$$

which implies $c'(0)t'(0)+v \in T_{x^0}M$.

Solution 3

(a) The map Φ is clearly C^1 hence differentiable. Its Jacobian matrix is given by

$$D\Phi(x) = \frac{1}{\|x\|^4}\begin{pmatrix} x_2^2 - x_1^2 & -2x_1x_2 \\ -2x_1x_2 & x_1^2 - x_2^2 \end{pmatrix}.$$

Thus,

$$|\det D\Phi(x)| = \frac{(x_1^2 - x_2^2)^2 + 4x_1x_2}{\|x\|^8} = \frac{(x_1^4 + x_2^4)}{\|x\|^8}.$$

It follows that $D\Phi(x) \in \mathrm{Aut}(\mathbb{R}^n)$ for all $x \neq 0$. It is clear that $\|\Phi(x)\| = \|x\|^{-1}$, from which it easily follows that that $\Phi(\Phi(x)) = x$, hence Φ is a bijection of $\mathbb{R}^2 \setminus \{0\}$ onto itself. Furthermore, Φ maps $B(0;1)\setminus\{0\} = \{x \in \mathbb{R}^2 \mid 0 < \|x\| < 1\}$ onto $\{x \in \mathbb{R}^2 \mid \|x\| > 1\} = \mathbb{R}^2 \setminus \bar{B}(0;1)$. By application of the change of variables theorem it follows that the function $(f\circ\Phi)|\det D\Phi|$ is absolutely integrable over $B(0;1)\setminus\{0\}$. Moreover the equality of (a) is valid.

(b) The function $N : x \mapsto x_1^4 + x_2^4$ is continuous hence attains a minimal value $m > 0$ on the unit sphere. By homogeneity, it follows that $N(tx) = t^4 N(x) \geq mt^4 = m\|tx\|^4$ for $\|x\| = 1$ and $t > 0$. This gives the required estimate.

(c) Assume that $g : x \mapsto \|x\|^{-s-2}$ is absolutely integrable over $\mathbb{R}^n \setminus \bar{B}(0;1)$. Let N be as in (b). Then it follows by substitution of variables that the function

$$|g \circ (\Phi(x))| N(x) \|x\|^{-8} = \|x\|^{s-6} N(x).$$

is absolutely integrable over $B(0;1) \setminus \{0\}$. Now $x \mapsto \|x\|^{-s-2}$ is continuous hence locally Riemann integrable over $B(0;1) \setminus \{0\}$. Furthemore, by (b) we have

$$\|x\|^{s-2} \leq m^{-1} \|x\|^{s-6} N(x).$$

Hence, for any Jordan measurable compact set $K \subset B(0;1) \setminus \{0\}$ we have

$$\int_K \|x\|^{s-2}\, dx \leq m^{-1} \int_{B(0;1)\setminus\{0\}} \|x\|^{s-6} N(x)\, dx < \infty.$$

The result now follows.

Solution 4

(a) We observe that $\text{div}(fv)(x) = \langle v(x), \text{grad} f(x) \rangle + f(x) \text{div} v(x)$ and that the vector field fv is C^1 on $U \supset \bar{\Omega}$. By Gauss' theorem, we have that the integral of this function over Ω equals

$$\int_{\partial\Omega} \langle f(y)v(y), \mathbf{n}(y) \rangle\, d_{n-1}y = \int_{\partial\Omega} \langle v(y), \mathbf{n}(y) \rangle\, g(y)\, d_{n-1}y.$$

This implies the result.

Another reasoning is as follows. The inner product $\langle v(x), \text{grad} f(x) \rangle$ is the sum of the scalar functions $v_j(x) D_j f(x)$, for $j = 1,2,3$. These are C^1 on $U \supset \bar{\Omega}$. By partial integration we have

$$\int_\Omega v_j(x) D_j f(x)\, dx = -\int_\Omega f(x)\, D_j v_j(x)\, dx + \int_{\partial\Omega} f(y) v_j(y) \mathbf{n}_j(y) d_{n-1}y.$$

By adding these equalities for $j = 1,2,3$ we obtain the required equality.

(b) First take $R > r$ sufficiently large such that $\text{supp}\, g$ is contained in the open ball $B(0;R)$. Then $\Omega := B(0;R) \setminus \bar{B}(r)$ is a bounded open domain with C^1-boundary.

The vector field w is C^1 on $U = \mathbb{R}^n \setminus \{0\}$ which contains the closure of Ω. Furthermore, $g|_U$ is C^1. It follows that we may apply (a).

We will first calculate $\text{div}\, w$. The partial derivative of $\|x\|^2$ with respect to x_j equals $2x_j$. Therefore,

$$\begin{aligned} D_j w(x) &= \frac{\partial}{\partial x_j} (x_j (\|x\|^2)^{-n/2} \\ &= \|x\|^{-n} + 2x_j^2 (-\frac{n}{2}) (\|x\|^2)^{-n/2-1} = \|x\|^n - n x_j^2 \|x\|^{-n-2}. \end{aligned}$$

Summing up over $1 \leq j \leq n$ we find

$$\operatorname{div} w(x) = n\|x\|^n - n\|x\|^2\|x\|^{-n-2} = 0.$$

Application of (a) now leads to

$$\int_\Omega \langle w(x)\,,\, \operatorname{grad} g(x)\rangle\, dx = \int_{\partial\Omega} g(y)\langle w(y)\,,\, \mathbf{n}(y)\rangle\, d_{n-1}y.$$

Now $\operatorname{grad} g = 0$ on $(\mathbb{R}^n \setminus \bar{B}(r)) \setminus \Omega$, hence the integral over Ω equals the similar integral over $\mathbb{R}^n \setminus \bar{B}(r)$. Furthermore, $\partial\Omega = S(r) \cup S(R)$ and $g = 0$ on $S(R)$, so that the integral over $\partial\Omega$ may be replaced by the integral over $S(r)$ without changing the outcome.

Now $w(y) = r^{-n}y$ for $y \in S(r)$, whereas the outward normal to $S(r)$ at y equals $\mathbf{n}(y) = -r^{-1}y$. Therefore,

$$\langle w(y)\,,\, \mathbf{n}(y)\rangle = -r^{-n-1}\|y\|^2 = -r^{-n+1} \qquad (y \in S(r)),$$

and we obtain the desired formula.