Retake Analyse in Meer Variabelen, WISB212 **2019-07-15, 9:00–12:00**

- Write your **name** on every sheet, and on the first sheet your **student number** and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Justify your anwers with complete arguments, unless specified otherwise. If you use results from the books or lecture notes, always **refer to them**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 40. The exam grade will be obtained from your total score through division by 4.
- You are free to write the solutions either in English, or in Dutch.

Good Luck !

10 pt total **Exercise 1.** In this exercise, we assume that $Y \in \mathrm{Mat}(n,\mathbb{R})$ is a fixed matrix.

1 pt (a) Show that the map $G : \mathrm{Mat}(n,\mathbb{R}) \to \mathrm{Mat}(n,\mathbb{R})$ defined by $X \mapsto YX$ is differentiable at every $X^0 \in \mathrm{Mat}(n,\mathbb{R})$, with derivative $DG(X^0) : \mathrm{Mat}(n,\mathbb{R}) \to \mathrm{Mat}(n,\mathbb{R})$ given by $DG(X^0)(H) = YH$.

5 pt (b) Let $U \subset \mathrm{Mat}(n,\mathbb{R})$ be an open neighborhood of the identity matrix I. Furthermore, let $f : U \to \mathrm{Mat}(n,\mathbb{R})$ be differentiable at I. Show that the map $g : U \to \mathrm{Mat}(n,\mathbb{R}), X \mapsto Xf(X)$, is differentiable at I with derivative given by

$$Dg(I) : \mathrm{Mat}(n,\mathbb{R}) \to \mathrm{Mat}(n,\mathbb{R}),\ H \mapsto Df(I)(H) + Hf(I).$$

In the following you may use that the map $F : \mathrm{GL(n,\mathbb{R})} \to \mathrm{Mat(n,\mathbb{R})}$, defined by $F(X) = X^{-1}YX$, is C^1. (This follows by application of Cramer's rule.)

4 pt (c) Show that the map F is differentiable at I and determine its derivative $DF(I) : \mathrm{Mat}(n,\mathbb{R}) \to \mathrm{Mat}(n,\mathbb{R})$

10 pt total **Exercise 2.** Let $M \subset \mathbb{R}^n$ be a C^1-submanifold of dimension $n-1$. We consider a C^1-curve $c : (-1,1) \to \mathbb{R}^n$ such that $c(0) = x^0 \in M$ and $c'(0) \notin T_{x_0}M$.

2 pt (a) Show that there exists an open neighborhood $U \ni x^0$ in $\mathbb{R}^n$ and a C^1-function $f : U \to \mathbb{R}$ with $Df(x^0) \neq 0$ and $M \cap U = f^{-1}(0)$.

2 pt (b) If f is as in (a), show that $(f \circ c)'(0) \neq 0$.

Let now $v \in \mathbb{R}^n$ be any vector.

4 pt (c) Show that there exist constants $\delta > 0$ and $\varepsilon > 0$ such that for every $s \in (-\delta, \delta)$ there exists a unique $t = t(s) \in (-\varepsilon, \varepsilon)$ such that $c(t) + sv \in M$.

2 pt (d) Show that the function $s \mapsto t(s)$ is differentiable at 0 and that

$$t'(0)c'(0) + v \in T_{x^0}M.$$

10 pt total **Exercise 3.** We consider the map $\Phi : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}^2 \setminus \{0\}$ given by

$$\Phi(x) = \frac{x}{\|x\|^2}.$$

5 pt (a) Let $B(0;1)$ denote the open unit ball in $\mathbb{R}^2$ and $\bar{B}(0;1)$ its closure. Show that for every absolutely Riemann integrable function $f : \mathbb{R}^2 \setminus \bar{B}(0;1) \to \mathbb{R}$ we have

$$\int_{\mathbb{R}^2 \setminus \bar{B}(0;1)} f(x)\, dx = \int_{B(0;1)\setminus\{0\}} \frac{x_1^4 + x_2^4}{\|x\|^8} f(\Phi(x))\, dx$$

2 pt (b) Show that there exists a constant $m > 0$ such that

$$x_1^4 + x_2^4 \geq m\|x\|^4$$

for all $x \in \mathbb{R}^2 \setminus \{0\}$. Hint: first consider $\|x\| = 1$.

3 pt (c) Let $s \in \mathbb{R}$. Show that if the function $x \mapsto \|x\|^{-s-2}$ is absolutely integrable over $\mathbb{R}^2 \setminus \bar{B}(0;1)$, then $x \mapsto \|x\|^{s-2}$ is absolutely integrable over $B(0;1) \setminus \{0\}$.

10 pt total **Exercise 4.** Let Ω be a bounded open domain with C^1-boundary in $\mathbb{R}^n$, and $U \subset \mathbb{R}^n$ an open set containing $\bar{\Omega}$. Furthermore, let $v : U \to \mathbb{R}^n$ be a C^1-vector field and $f \in C^1(U)$. Let $\mathbf{n}$ be the outward unit normal on $\partial\Omega$.

4 pt (a) Show that

$$(*) \quad \int_\Omega \langle v(x)\,, \operatorname{grad} f(x)\rangle\, dx = -\int_\Omega f(x) \operatorname{div} v(x)\, dx + \int_{\partial\Omega} \langle v(y)\,, \mathbf{n}(y)\rangle\, f(y)\, d_{n-1}y.$$

In the following we assume that $n \geq 2$, and that $w(x) := \|x\|^{-n}x$. For $r > 0$ we write $\bar{B}(r) := \{x \in \mathbb{R}^n \mid \|x\| \leq r\}$ and $S(r) := \{x \in \mathbb{R}^n \mid \|x\| = r\}$.

6 pt (b) Show that for every $g \in C_c^1(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n \setminus \bar{B}(r)} \langle w(x)\,, \operatorname{grad} g(x)\rangle\, dx = -r^{1-n} \int_{S(r)} g(y)\, d_{n-1}y.$$

Hint: replace the domain of integration by a suitable bounded domain.