## Inleiding Topologie, Exam A (April 18, 2012)

Exercise 1. Let $\mathcal{B}$ be the familly of subsets of $\mathbb{R}$ consisting of $\mathbb{R}$ and the subsets

$$
[n, a):=\{r \in \mathbb{R}: n \leq r<a\} \quad \text { with } n \in \mathbb{Z}, a \in \mathbb{R}
$$

1. Show that $\mathcal{B}$ is not a topology on $\mathbb{R}$, but it is a topology basis. Denote by $\mathcal{T}$ the associated topology. (1p)
2. Is $(\mathbb{R}, \mathcal{T})$ second countable? But Hausdorff? But metrizable? Can it be embedded in $\mathbb{R}^{2012}$ (with the Euclidean topology)? (1p)
3. compute the closure, the interior and the boundary of $A=\left[-\frac{1}{2}, \frac{1}{2}\right]$ in $(\mathbb{R}, \mathcal{T})$. (1.5p)

Exercise 2. Prove directly that the abstract torus $T_{\text {abs }}$ is homeomorphic to $S^{1} \times S^{1}$. More precisely, define an explicit map

$$
\tilde{f}:[0,1] \times[0,1] \rightarrow \mathbb{R}^{4}
$$

whose image is

$$
S^{1} \times S^{1}=\left\{(x, y, z, t) \in \mathbb{R}^{4}: x^{2}+y^{2}=z^{2}+t^{2}=1\right\}
$$

and which induces a homeomorphism $f: T_{\mathrm{abs}} \rightarrow \mathbb{S}^{1} \times S^{1}$. Provide all the arguments. (1.5p).

Exercise 3. Let $X$ be the space obtained from the sphere $S^{2}$ by gluing the north and the south pole (with the quotient topology). Show that $X$ can be obtained from a square $[0,1] \times[0,1]$ by glueing some of the points on the boundary (note: you are not allowed to glue a point in the interior of the square to any other point). More precisely:

1. Describe the equivalence relation $R_{0}$ on $S^{2}$ encoding the glueing that defines $X$. (0.25p)
2. Make a picture of $X$ in $\mathbb{R}^{3}$. (0.25p)
3. Describe an equivalence relation $R$ on $[0,1] \times[0,1]$ encoding a glueing with the required properties. ( $1 p$ )
4. Show that, indeed, $X$ is homeomorphic to $[0,1] \times[0,1] / R$ (provide as many arguments as you can, but do not write down explicit maps- instead, indicate them on the picture). (0.5p)

Exercise 4. Show that:

1. There exist continuous surjective maps $f: S^{1} \rightarrow S^{1}$ which are not injective. (0.5p)
2. Any continuous injective map $f: S^{1} \rightarrow S^{1}$ is surjective. (1p)

Exercise 5. Show that any continuous map

$$
f:\left(\mathbb{R}, \mathcal{T}_{\text {Eucl }}\right) \rightarrow\left(\mathbb{R}, \mathcal{T}_{l}\right)
$$

must be constant (recall that $\mathcal{T}_{l}$ is the lower limit topology- i.e. the one generated by intervals of type $[a, b))$. (1.5p)

Note 1: Motivate all your answers. Whenever you use a Theorem or Proposition, please make that clear (e.g. by stating it). Please write clearly (English or Dutch).

Note 2: The final mark is

$$
\min \{10,1+p\},
$$

where $p$ is the number of points you collect from the exercises.

