## Inleiding Topologie, Exam A (April 18, 2012)

**Exercise 1.** Let  $\mathcal{B}$  be the familly of subsets of  $\mathbb{R}$  consisting of  $\mathbb{R}$  and the subsets

$$[n, a) := \{ r \in \mathbb{R} : n \le r < a \} \quad \text{with} \quad n \in \mathbb{Z}, a \in \mathbb{R}.$$

- 1. Show that  $\mathcal{B}$  is not a topology on  $\mathbb{R}$ , but it is a topology basis. Denote by  $\mathcal{T}$  the associated topology. (1p)
- 2. Is  $(\mathbb{R}, \mathcal{T})$  second countable? But Hausdorff? But metrizable? Can it be embedded in  $\mathbb{R}^{2012}$  (with the Euclidean topology)? (1p)
- 3. compute the closure, the interior and the boundary of  $A = \left[-\frac{1}{2}, \frac{1}{2}\right]$  in  $(\mathbb{R}, \mathcal{T})$ . (1.5p)

**Exercise 2.** Prove directly that the abstract torus  $T_{abs}$  is homeomorphic to  $S^1 \times S^1$ . More precisely, define an explicit map

$$\tilde{f}: [0,1] \times [0,1] \to \mathbb{R}^4$$

whose image is

$$S^1 \times S^1 = \{(x, y, z, t) \in \mathbb{R}^4 : x^2 + y^2 = z^2 + t^2 = 1\}$$

and which induces a homeomorphism  $f: T_{abs} \to \mathbb{S}^1 \times S^1$ . Provide all the arguments. (1.5p).

**Exercise 3.** Let X be the space obtained from the sphere  $S^2$  by gluing the north and the south pole (with the quotient topology). Show that X can be obtained from a square  $[0,1] \times [0,1]$  by glueing some of the points on the *boundary* (note: you are not allowed to glue a point in the *interior* of the square to any other point). More precisely:

- 1. Describe the equivalence relation  $R_0$  on  $S^2$  encoding the glueing that defines X. (0.25p)
- 2. Make a picture of X in  $\mathbb{R}^3$ . (0.25p)
- 3. Describe an equivalence relation R on  $[0,1] \times [0,1]$  encoding a glueing with the required properties. (1p)
- 4. Show that, indeed, X is homeomorphic to  $[0,1] \times [0,1]/R$  (provide as many arguments as you can, but do not write down explicit maps- instead, indicate them on the picture). (0.5p)

Exercise 4. Show that:

- 1. There exist continuous surjective maps  $f: S^1 \to S^1$  which are not injective. (0.5p)
- 2. Any continuous injective map  $f: S^1 \to S^1$  is surjective. (1p)

Exercise 5. Show that any continuous map

$$f: (\mathbb{R}, \mathcal{T}_{\mathrm{Eucl}}) \to (\mathbb{R}, \mathcal{T}_{l})$$

must be constant (recall that  $\mathcal{T}_l$  is the lower limit topology- i.e. the one generated by intervals of type [a, b)). (1.5p)

Note 1: Motivate all your answers. Whenever you use a Theorem or Proposition, please make that clear (e.g. by stating it). Please write clearly (English or Dutch).

Note 2: The final mark is

$$\min\{10, 1+p\},\$$

where p is the number of points you collect from the exercises.