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Uitwerkingen Hertentamen Inleiding Financiele Wiskunde, 2011-12

* Punten per opgave:

opgave:	1	2	3	4
punten:	30	20	20	30

1. Consider a 2-period binomial model with $S_0 = 20$, $u = 1.3$, $d = 0.9$, and $r = 0.1$. Suppose the real probability measure P satisfies $P(H) = p = \frac{1}{3} = 1 - P(T)$.

   (a) Consider an Asian European option with payoff $V_2 = ((S_1 + S_2)/2 - 20)^+$. Determine the price V_n at time $n = 0, 1$.

   (b) Suppose $\omega_1\omega_2 = HT$, find the values of the portfolio process $\Delta_0, \Delta_1(H)$ so that the corresponding wealth process satisfies $X_0 = V_0$ (your answer in part (a)) and $X_2(HT) = V_2(HT)$.

   (c) Consider the utility function $U(x) = 4x^{1/4}$ $(x > 0)$. Show that the random variable $X = X_2$ (which is a function of the two coin tosses) that maximizes $E(U(X))$ subject to the condition that $\widetilde{E}\left(\frac{X}{(1+r)^2}\right) = X_0$ is given by

   $$X = X_2 = \frac{(1.1)^2 X_0}{Z^{4/3} E(Z^{-1/3})}.$$

   (d) Consider part (c) and assume $X_0 = 20$. Determine the value of the optimal portfolio process $\{\Delta_0, \Delta_1\}$ and the value of the corresponding wealth process $\{X_0, X_1, X_2\}$.

   (e) Consider now an Asian American put option with expiration $N = 2$, and intrinsic value $G_n = 20 - \dfrac{S_0 + \cdots + S_n}{n+1}$, $n = 0, 1, 2$. Determine the price V_n at time $n = 0, 1$ of the American option. Find the optimal exercise time $\tau^*(\omega_1\omega_2)$ for all $\omega_1\omega_2$.

Solution (a): We first calculate the risk-neutral probability measure $\widetilde{P}$, we have $\widetilde{P}(H) = \widetilde{p} = 1/2$ and $\widetilde{P}(T) = \widetilde{q} = 1/2$. We start with the value of V_2, we have $V_2(HH) = 9.9, V_2(HT) = 4.7, V_2(TH) = 0.7, V_2(TT) = 0$. Then

$$V_1(H) = \frac{1}{1.1}[\frac{1}{2}(9.9) + \frac{1}{2}(4.7)] = 6.64,$$

and

$$V_1(T) = \frac{1}{1.1}[\frac{1}{2}(0.7) + \frac{1}{2}(0)] = 0.32,$$

leading to

$$V_0 = \frac{1}{1.1}[\frac{1}{2}(6.64) + \frac{1}{2}(0.32)] = 3.16.$$

Solution (b): If $\omega_1\omega_2 = HT$, then

$$\Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{6.64 - 0.32}{26 - 18} = 0.79,$$

and

$$\Delta_1(H) = \frac{V_1(HH) - V_1(HT)}{S_1(HH) - S_1(HT)} = \frac{9.94 - 4.7}{33.8 - 23.4} = 0.5.$$

Leading to

$$X_1(H) = \Delta_0 S1_(T) + 1.1(V_0 - \Delta_0 S_0) = 6.64,$$

and

$$X_2(TH) = \Delta_1(T)S_2(TH) + 1.1(X_1(T) - \Delta_1(T)S_1(T)) = 4.7.$$

Solution (d): Notice that the function $U(x) = 4x^{1/4}$, $x > 0$ is strict concave with $U'(x) = \frac{1}{2\sqrt{x}}$. We apply Theorem 3.3.6, we find that the inverse I of U' is given by $I(x) = x^{-4/3}$. Thus, the optimal solution is given by

$$X_2 = X = I\left(\frac{\lambda Z}{(1.1)^2}\right) = \frac{(1.1)^{8/3}}{\lambda^{4/3}Z^{4/3}},$$

and satisfies the constraint

$$X_0 = E\left(\frac{XZ}{(1.1)^2}\right) = \frac{(1.1)^{2/3}}{\lambda^{4/3}}E\left(Z^{-1/3}\right).$$

Hence, $\lambda^{4/3} = \frac{(1.1)^{2/3}E(Z^{-1/3})}{X_0}$, and

$$X = \frac{X_0(1.1)^2}{Z^{4/3}E(Z^{-1/3})}.$$

Solution (e): To find the optimal portfolio and corresponding wealth processes, we first determine explicitly the the random variable $X = X_2$, and then we apply Theorem 1.2.2 with $X_0 = 20$. We begin by find the Radon Nikodym derivative Z. We have

$$Z(HH) = \frac{9}{4},\ Z(HT) = Z(TH) = \frac{9}{8},\ Z(TT) = \frac{9}{16}.$$

Next, we find

$$E(Z^{-1/3}) = (\frac{4}{9})^{1/3} \times \frac{1}{9} + (\frac{8}{9})^{1/3} \times \frac{2}{9} + (\frac{8}{9})^{1/3} \times \frac{2}{9} + (\frac{16}{9})^{1/3} \times \frac{4}{9} = 1.05.$$

Thus,

$$X = X_2 = \frac{X_0(1.1)^2}{Z^{4/3}E(Z^{-1/3})} = \frac{23.05}{Z^{4/3}}.$$

This leads to

$$X_2(HH) = 7.81,\ X_2(HT) = X_2(TH) = 19.70,\ X_2(TT) = 50.11.$$

Hence,

$$X_1(H) = \frac{1}{1.1}[\frac{1}{2}(7.81) + \frac{1}{2}(19.70)] = 12.50,$$
$$X_1(T) = \frac{1}{1.1}[\frac{1}{2}(19.70) + \frac{1}{2}(50.11)] = 31.13.$$

Notice that

$$X_0 = \frac{1}{1.1}[\frac{1}{2}(12.50) + \frac{1}{2}(31.73)] = 20.1$$

as required. The optimal portfolio is given by

$$\Delta_0 = \frac{X_1(H) - X_1(T)}{S_1(H) - S_1(T)} = \frac{12.50 - 31.73}{26 - 18} = -2.40,$$

$$\Delta_(H) = \frac{X_2(HH) - X_2(HT)}{S_2(HH) - S_2(HT)} = \frac{7.81 - 19.70}{33.8 - 23.4} = -1.14,$$

$$\Delta_1(T) = \frac{X_2(TH) - X_2(TT)}{S_2(TH) - S_2T(T)} = \frac{19.70 - 50.11}{23.4 - 16.2} = -4.22.$$

Solution (e): The intrinsic value process is given by

$$G_0 = 0, G_1(H) = -3, G_1(T) = 1,$$

$$G_2(HH) = -6.6, G_2(HT) = -3.13, G_2(TH) = -0.47, G_2(TT) = 1.93.$$

The payoff at time 2 is given by

$$V_2(HH) = V_2(HT) = V_2(TH) = 0,\ V_2(TT) = 1.93.$$

Applying the American algorithm, we get

$$V_1(H) = \max\left(-3, \frac{1}{1.1}[\frac{1}{2}\times 0 + \frac{1}{2}\times 0]\right) = 0.$$

$$V_1(T) = \max\left(1, \frac{1}{1.1}[\frac{1}{2}\times 0 + \frac{1}{2}\times 1.93]\right) = \max(1, 0.88) = 1.$$

$$V_0 = \max\left(0, \frac{1}{1.1}[\frac{1}{2}\times 0 + \frac{1}{2}\times 1]\right) = \max(0, 0.455) = 0.455.$$

The optimal exercise time is given by

$$\tau^*(HH) = \tau^*(HT) = \infty,\ \tau^*(TH) = \tau^*(TT) = 1.$$

2. Consider a 3-period (non constant interest rate) binomial model with interest rate process R_0, R_1, R_2 defined by

$$R_0 = 0,\ R_1(\omega_1) = .05 + .01H_1(\omega_1),\ R_2(\omega_1,\omega_2) = .05 + .01H_2(\omega_1,\omega_2)$$

where $H_i(\omega_1,\cdots,\omega_i)$ equals the number of heads appearing in the first i coin tosses $\omega_1,\cdots,\omega_i$. Suppose that the risk neutral measure is given by $\widetilde{P}(HHH) = \widetilde{P}(HHT) = 1/8$, $\widetilde{P}(HTH) = \widetilde{P}(THH) = \widetilde{P}(THT) = 1/12$, $\widetilde{P}(HTT) = 1/6$, $\widetilde{P}(TTH) = 1/9$, $\widetilde{P}(TTT) = 2/9$.

(a) Calculate $B_{1,2}$ and $B_{1,3}$, the time one price of a zero coupon maturing at time two and three respectively.

(b) Consider a 3-period interest rate swap. Find the 3-period swap rate SR_3, i.e. the value of K that makes the time zero no arbitrage price of the swap equal to zero.

(c) Consider a 3-period floor that makes payments $F_n = (.055 - R_{n-1})^+$ at time $n = 1,2,3$. Find Floor_3, the price of this floor.

Solution (a): We first calcultate the values of R_0, R_1, R_2 and D_1, D_2, D_3 in the following tables:

$\omega_1\omega_2$	R_0	R_1	R_2
HH	0	0.06	0.07
HT	0	0.06	0.06
TH	0	0.05	0.06
TT	0	0.05	0.05

$\omega_1\omega_2$	$\frac{1}{1+R_0}$	$\frac{1}{1+R_1}$	$\frac{1}{1+R_2}$	D_1	D_2	D_3	$\widetilde{P}$
HH	1	$\frac{1}{1.06}$	$\frac{1}{1.07}$	1	$\frac{1}{1.06}$	$\frac{1}{1.1342}$	$\frac{1}{4}$
HT	1	$\frac{1}{1.06}$	$\frac{1}{1.06}$	1	$\frac{1}{1.06}$	$\frac{1}{1.1236}$	$\frac{1}{4}$
TH	1	$\frac{1}{1.05}$	$\frac{1}{1.06}$	1	$\frac{1}{1.05}$	$\frac{1}{1.113}$	$\frac{1}{6}$
TT	1	$\frac{1}{1.05}$	$\frac{1}{1.05}$	1	$\frac{1}{1.05}$	$\frac{1}{1.1025}$	$\frac{1}{3}$

Since $D_1 = 1$ and D_2 is known at time 1, then $B_{1,2} = \widetilde{E}_1(D_2) = D_2$. This gives $B_{1,2}(H) = 1/1.06$ and $B_{1,2}(T) = 1/1.05$.

Now, D_3 depends on the first two coin tosses only, and since $D_1 = 1$ we have

$$\begin{aligned} B_{1,3}(H) = \widetilde{E}_1(D_3)(H) &= D_3(HH)\widetilde{P}(\omega_2 = H|\omega_1 = H) + D_3(HT)\widetilde{P}(\omega_2 = T|\omega_1 = H) \\ &= \frac{1}{1.1342}\frac{1}{2} + \frac{1}{1.1236}\frac{1}{2} = 0.8858, \end{aligned}$$

and

$$\begin{aligned} B_{1,3}(T) = \widetilde{E}_1(D_3)(T) &= D_3(TH)\widetilde{P}(\omega_2 = H|\omega_1 = T) + D_3(TT)\widetilde{P}(\omega_2 = T|\omega_1 = T) \\ &= \frac{1}{1.113}\frac{1}{3} + \frac{1}{1.1025}\frac{2}{3} = 0.9499. \end{aligned}$$

Solution (b): From Theorem 6.3.7, we know that

$$SR_3 = \frac{1 - B_{0,3}}{B_{0,1} + B_{0,2} + B_{0,3}}.$$

Now,

$$B_{0,1} = \widetilde{E}(D_1) = 1,$$

D_2 depends on the ω_1 only, so

$$\begin{aligned} B_{0,2} = \widetilde{E}(D_2) &= \frac{1}{1.06}\,\widetilde{P}(\omega_1 = H) + \frac{1}{1.05}\,\widetilde{P}(\omega_1 = T) \\ &= \frac{1}{1.06}\frac{1}{2} + \frac{1}{1.05}\frac{1}{2} = 0.94789, \end{aligned}$$

Now, D_3 depends only on ω_1, ω_2, hence

$$\begin{aligned} B_{0,3} = \widetilde{E}(D_3) &= \frac{1}{1.1342}\,\widetilde{P}(\omega_1 = H, \omega_2 = H) + \frac{1}{1.1236}\,\widetilde{P}(\omega_1 = H, \omega_2 = T) \\ &+ \frac{1}{1.113}\,\widetilde{P}(\omega_1 = T, \omega_2 = H) + \frac{1}{1.1025}\,\widetilde{P}(\omega_1 = H, \omega_2 = H) \\ &= \frac{1}{1.1342}\frac{1}{4} + \frac{1}{1.1236}\frac{1}{4} + \frac{1}{1.113}\frac{1}{6} + \frac{1}{1.1025}\frac{1}{3} \\ &= 0.895. \end{aligned}$$

Thus,

$$SR_3 = \frac{1 - B_{0,3}}{B_{0,1} + B_{0,2} + B_{0,3}} = \frac{1 - 0.91787}{2.86576} = 0.0287.$$

Solution (c): From Definition 6.3.8 we have

$$Floor_3 = \sum_{n=1}^{3} \widetilde{E}(D_n(0.055 - R_{n-1})^+ .)$$

We display the values of $(0.055 - R_{n-1})^+)$ in a table

$\omega_1\omega_2$	$(0.055 - R_0)^+$	$(0.055 - R_1)^+$	$(0.055 - R_2)^+$
HH	0.055	0	0
HT	0.055	0	0
TH	0.055	0.005	0
TT	0.055	0.005	0.005

Thus,

$$\widetilde{E}(D_1(0.055 - R_0)^+) = 0.055,$$

$$\widetilde{E}(D_2(0.055 - R_1)^+) = D_2(H)(0)P(H) + D_2(T)(0.005)P(T) = \frac{1}{1.05}(0.005)\frac{1}{2} = 0.00238,$$

and

$$\widetilde{E}(D_3(0.055 - R_2)^+) = D_3(TT)(0.055)P(TT) = \frac{1}{1.1025}(0.005)\frac{1}{3} = 0.00151$$

Therefore,

$$Floor_3 = 0.055 + 0.00238 + 0.00151 = 0.05889.$$

3. Consider the binomial model with $u = 2^1$, $d = 2^{-1}$, and $r = 1/4$, and consider a perpetual American put option with $S_0 = 10$ and $K = 12$. Suppose that Alice and Bob each buy such an option

   (a) Suppose that Alice uses the strategy of exercising the first time the price reaches 5 euros. What should then the price be at time 0?

   (b) Suppose that Bob uses the strategy of exercising the first time the price reaches 2.5 euros. What should then the price be at time 0?

   (c) What is the probability that the price reaches 20 euros for the first time at time $n = 5$?

   Solution (a): The buyer is using the exercise policy τ_{-2}. Hence, the price at time 0 should be

$$V_0 = V^{\tau_{-2}} = \widetilde{E}\left(\left(\frac{4}{5}\right)^{\tau_{-2}} (24 - S_{\tau_{-2}})\right) = (\frac{1}{2})^2(24-5) = 4.75.$$

   Solution (b): The buyer is using the exercise policy τ_{-4}. Hence, the price at time 0 should be

$$V_0 = V^{\tau_{-4}} = \widetilde{E}\left(\left(\frac{4}{5}\right)^{\tau_{-4}} (24 - S_{\tau_{-4}})\right) = (\frac{1}{2})^4(24-1.25) = 1.42.$$

   Solution (c): The probability that the price reaches 80 for the first time at time 5 is equal to $P(\{\tau_2 = 5\}) = 0$ since it is impossible for the random walk to reach level 2 after an odd number of steps.

4. Consider a random walk $M_0, M_1, \cdots$ with probability p for an up step and $q = 1-p$ for a down step, $0 < p < 1$. For $a \in \mathbb{R}$ and $b > 1$, define $S_n^a = b^{-n}2^{aM_n}$, $n = 0, 1, 2, \cdots$.

   (a) For which values of a is the the process $S_0^a, S_1^a, \cdots$ a (i) martingale, (ii) supermartingale, (iii) submartingale?

   (b) Show that the process $S_0^a, S_1^a, \cdots$ is a Markov Process.

   (c) Suppose now that $p = 1/2$, so $M_0, M_1, \cdots$, is the symmetric random walk. Let $\tau_m = \inf\{n \geq 0 : M_n = m\}$. Determine the value of $E(S_{\tau_m}^a)$.

   Solution (a): First note that the process (S_n^a) is adjusted, and

$$S_{n+1}^a = b^{-n-1}\, 2^{aM_n + aX_{n+1}} = S_n^a\, b^{-1} 2^{aX_{n+1}}.$$

   Since X_{n+1} is independent of the first n tosses we have

$$E_n(2^{aX_{n+1}}) = E(2^{aX_{n+1}}) = 2^a\, p + 2^{-a}\, q.$$

Thus,

$$E_n(S_{n+1}^a) = S_n^a b^{-1}(2^a\, p + 2^{-a}\, q).$$

(i) For the process to be a martingale, we need to find the values of a such that

$$b^{-1}(2^a\, p + 2^{-a}\, q) = 1$$

or equivalently,

$$p2^{2a} - b\, 10^a + q = 0.$$

Solving, we get

$$2^a = \frac{b \pm \sqrt{b^2 - 4pq}}{2p}$$

implying

$$a = \log_2 \left(\frac{b \pm \sqrt{b^2 - 4pq}}{2p} \right).$$

(ii) For the process to be a submartingale, we need to find the values of a such that

$$b^{-1}(2^a\, p + 2^{-a}\, q) \leq 1$$

or equivalently,

$$p2^{2a} - b\, 10^a + q \leq 0.$$

Solving, we get

$$2^a \leq \frac{b - \sqrt{b^2 - 4pq}}{2p}, \quad \text{or } 2^a \geq \frac{b + \sqrt{b^2 - 4pq}}{2p}$$

implying

$$a \leq \log_2 \left(\frac{b - \sqrt{b^2 - 4pq}}{2p} \right) \quad \text{or } a \geq \log_2 \left(\frac{b\sqrt{b^2 - 4pq}}{2p} \right).$$

(iii) For the process to be a supermartingale, we need to find the values of a such that

$$b^{-1}(2^a\, p + 2^{-a}\, q) \geq 1$$

or equivalently,

$$p2^{2a} - b\, 10^a + q \geq 0.$$

Solving, we get

$$\frac{b - \sqrt{b^2 - 4pq}}{2p} \leq 2^a \leq \frac{b + \sqrt{b^2 - 4pq}}{2p}$$

implying

$$\log_2 \left(\frac{b - \sqrt{b^2 - 4pq}}{2p} \right) \leq a \leq \log_2 \left(\frac{b\sqrt{b^2 - 4pq}}{2p} \right).$$

Solution (b): Note that $S_{n+1}^a = S_n^a\, b^{-1} 2^{aX_{n+1}}$. Let f be any real function, define a function F on $\mathbb{R}^2$ by $F(s,x) = f(sb^{-1}2^{ax})$. Then, $F(S_n^a, X_{n+1}) = f(S_{n+1}^a)$. Notice that S_n^a depends on the first n tosses while X_{n+1} is independent of the first n tosses. By the Independence Lemma, we have

$$E_n(f(S_{n+1}^a)) = E_n(F(S_n^a, X_{n+1})) = g(S_n^a),$$

where

$$g(s) = E(F(s, X_{n+1})) = E(f(sb^{-1}2^{aX_{n+1}})) = pf(sb^{-1}2^a) + qf(sb^{-1}2^{-a}).$$

In particular,

$$g(S_n^a) = pf(S_n^a b^{-1}2^a) + qf(S_n^a b^{-1}2^{-a}).$$

Thus, (S_n^a) is a Markov Process.

Solution (c): Observe that $S_{\tau_m}^a = b^{-\tau_m}2^{aM_{\tau_m}} = b^{-\tau_m}2^{ma}$. By Theorem 5.2.3 we have

$$E(S_{\tau_m}^a) = 2^{ma}E(b^{\tau_n}) = b^{ma}(\frac{1-\sqrt{1-b^2}}{b})^m.$$