Differentiable manifolds – Exam 1

Notes:

1. **Write your name and student number *****clearly*** **on each page of written solutions you hand in.**
2. You can give solutions in English or Dutch.
3. You are expected to explain your answers.
4. You are allowed to consult text books and class notes.
5. You are **not** allowed to consult colleagues, calculators, computers etc.
6. Advice: read all questions first, then start solving the ones you already know how to solve or have good idea on the steps to find a solution. After you have finished the ones you found easier, tackle the harder ones.

Some definitions you should know, but may have forgotten

- The *quaternions*, $\mathbb{H}$, are isomorphic to $\mathbb{R}^4$ as vector space and are endowed with a multiplication which makes $\mathbb{H}\backslash\{0\}$ into a group. This multiplication is $\mathbb{R}$-bilinear and is defined on a basis $\{1, i, j, k\}$ of $\mathbb{H}$ by
$$i^2 = j^2 = k^2 = ijk = -1.$$
- A *Lie group* is a differentiable manifold G endowed with the structure of a group and for which the maps
$$\begin{array}{rl} G \times G \longrightarrow G & \quad (g,h) \mapsto g \cdot h \\ G \longrightarrow G & \quad g \mapsto g^{-1} \end{array}$$
are smooth.

Questions

1) (1.5 pt) Let $E \stackrel{\pi}{\longrightarrow} M$ be a vector bundle over a manifold M and let $s : M \longrightarrow E$ be a section. Show that s is an embedding of M on E.

2) Let M be the subset of $\mathbb{R}^3$ defined by the equation
$$M = \{(x_1, x_2, x_3) : x_1^3 + x_2^3 + x_3^3 + 3x_1x_2x_3 = 1\}.$$

a) (1 pt) Show that M is an embedded submanifold of $\mathbb{R}^3$;

b) (1.5 pt) Define $\pi : M \longrightarrow \mathbb{R}$; $\pi(x_1, x_2, x_3) = x_1$. Find the critical points and critical values of π.

3) (2 pt) Let $\varphi : M \longrightarrow N$ be an embedding such that $\varphi(M)$ is a closed subset of N. Let $X \in \Gamma(TM)$ be a vector field. Show that there is a vector field $Y \in \Gamma(TN)$ such that $\varphi_*(X|_p) = Y|_{\varphi(p)}$.

4) Let $\mathbb{H}$ be the space of quaternions.

a) (1 pt) Show that $\mathbb{H}\backslash\{0\}$ is a Lie group if endowed with quaternionic multiplication as group operation.

b) (1 pt) Show that the 3-dimensional sphere $S^3 \subset \mathbb{H}\backslash\{0\}$ is also a Lie group with quaternionic multiplication as group operation.

5) (2 pt) Let $\mathfrak{U} = \{U_\alpha : \alpha \in A\}$ be a cover of a manifold M for which each open set $U_{\alpha_0 \cdots \alpha_n}$ is either empty or homeomorphic to a disc. Show that $\check{H}^k(M; \mathbb{Z}_2; \mathfrak{U}) \cong \check{H}^k(M; \mathbb{C}^\infty(M; \mathbb{R}^*); \mathfrak{U})$ for all $k \in \mathbb{Z}$.

Hint: A possible approach using the sequence

$$\mathbb{Z}_2 \lhd C^\infty(U; \mathbb{R}^*) \longrightarrow C^\infty(U; \mathbb{R}).$$

a) Consider $\mathbb{Z}_2 \subset \mathbb{R}^*$ as the set $\{1, -1\}$. The inclusion $\iota : \mathbb{Z}_2 \hookrightarrow C^\infty(M; \mathbb{R}^*)$ gives rise to a map of cochains

$$\iota : \check{C}^k(M; \mathbb{Z}_2; \mathfrak{U}) \longrightarrow \check{C}^k(M; C^\infty(M; \mathbb{R}^*); \mathfrak{U})$$

Show that ι commutes with Čech differentials and hence induces a map in cohomology:

$$\iota^* : \check{H}^k(M; \mathbb{Z}_2; \mathfrak{U}) \longrightarrow \check{H}^k(M; \mathbb{C}^\infty(M; \mathbb{R}^*); \mathfrak{U}).$$

b) Consider the map

$$r : C^\infty(M; \mathbb{R}^*) \hookrightarrow \mathbb{Z}_2; \qquad r(f) = \frac{f}{|f|}$$

Show that r commutes with Čech differentials and hence induces a map in cohomology.

$$r^* : \check{H}^k(M; \mathbb{C}^\infty(M; \mathbb{R}^*); \mathfrak{U}) \longrightarrow \check{H}^k(M; \mathbb{Z}_2; \mathfrak{U}).$$

c) Show that r^* is a right and left inverse for ι^*.