

# Functionaalanalyse, WISB315

## Hertentamen

Family name: \_\_\_\_\_ Given name: \_\_\_\_\_

Student number: \_\_\_\_\_

**Please:**

- Switch off your mobile phone and put it into your bag.
- Write with a blue or black pen, not with a green or red one, nor with a pencil.
- Write your name on each sheet.
- Hand in this sheet, as well.
- Hand in only one solution to each problem.

The examination time is 180 minutes.

You are not allowed to use books, calculators, or lecture notes, but you may use 1 sheet of handwritten personal notes (A4, both sides).

Unless otherwise stated, you may use results that were proved in the lecture or in the book by Rynne and Youngson, without proving them.

Prove every other statement you make. Justify your calculations. Check the hypotheses of the theorems you use.

You may write in Dutch.

27 points will yield a passing grade 6, and 56 points a grade 10.

Good luck and Merry Christmas!

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/4	/5	/7	/6	/5	/6	/6	/6	/16	/61



**Problem 1** (basis and inner product, 4 pt). Let  $X$  be a finite-dimensional  $\mathbb{C}$ -vector space and  $v_1, \dots, v_n$  a basis of  $X$ . For  $x \in X$  we denote by  $x^1, \dots, x^n \in \mathbb{C}$  the coordinates of  $x$  with respect to  $v_1, \dots, v_n$ . (This is the unique collection of numbers satisfying the equality  $x = \sum_{i=1}^n x^i v_i$ .) Show that the map

$$\langle \cdot, \cdot \rangle : X \times X \rightarrow \mathbb{C}, \quad \langle x, y \rangle := \sum_{i=1}^n x^i \bar{y}^i,$$

is an inner product.

**Remark:** You do not need to prove that the coordinates of  $x$  are well-defined. You may also use that they depend on  $x$  in a linear way.

**Problem 2** (quotient norm, 5 pt). Let  $(X, \|\cdot\|)$  be a normed vector space and  $Y \subseteq X$  a linear subspace. We define

$$\begin{aligned} \|\cdot\|^Y : X/Y &\rightarrow [0, \infty), \\ \|\tilde{x}\|^Y &:= \inf_{y \in Y} \|x - y\|, \end{aligned} \tag{1}$$

where  $x \in \tilde{x}$  is an arbitrary representative of  $\tilde{x}$ . Prove the following:

- (i) The map  $\|\cdot\|^Y$  is well-defined, i.e., the right hand side of (1) does not depend on the choice of  $x$ .
- (ii) If  $Y$  is closed, then  $\|\cdot\|^Y$  is nondegenerate.

**Remark:**  $X/Y$  is a vector space and  $\|\cdot\|^Y$  is a seminorm. You do not need to prove these facts.

**Problem 3** ( $\ell^p$  is complete, 7 pt). Let  $p \in [1, \infty)$ . Show that the norm  $\|\cdot\|_p$  on  $\ell^p = \ell^p(\mathbb{N})$  is complete.

**Remark:** You do not need to show that  $\ell^p$  is a vector space nor that  $\|\cdot\|_p$  is a norm.

**Problem 4** (bounded linear map on  $\ell^p$ , 6 pt). Let

$$p \in (1, \infty), \quad y \in \ell^{\frac{p}{p-1}}.$$

Show that the operator

$$T : \ell^p \rightarrow \mathbb{K}, \quad Tx := \sum_{i=1}^{\infty} x^i y^i,$$

is bounded and calculate its operator norm.

**Remark:** You do not need to prove that  $T$  is well-defined.



**Problem 5** (Closed Graph Theorem, 5 pt). Let  $X, Y$  be Banach spaces and  $T : X \rightarrow Y$  a linear map, such that the graph of  $T$  is closed (with respect to the product topology). Show that  $T$  is bounded.

**Remarks:** This was a corollary in the lecture. You need to prove this corollary here.

You may use the fact that the map

$$\| \cdot \|_1 : X \times Y \rightarrow [0, \infty), \quad \|(x, y)\|_1 := \|x\|_X + \|y\|_Y,$$

is a complete norm.

**Hint:** Use a theorem from the lecture.

**Problem 6** (properties of adjoint, 6 pt). Let  $H_1, H_2$  be Hilbert spaces and  $T \in B(H_1, H_2)$ . Recall that the adjoint of  $T$  is defined to be the map

$$T^* := \Phi_{H_1}^{-1} T' \Phi_{H_2} : H_2 \rightarrow H_1,$$

where

$$\Phi_H : H \rightarrow H', \quad \Phi_H(x) := \langle \cdot, x \rangle.$$

Prove the following:

(i) For every  $x_2 \in H_2$  we have

$$\langle y_1, T^* x_2 \rangle_{H_1} = \langle T y_1, x_2 \rangle_{H_2}, \quad \forall y_1 \in H_1, \quad (2)$$

and  $T^* x_2$  is uniquely determined by this equation.

(ii)

$$(T^*)^* = T. \quad (3)$$

**Problem 7** ( $\ell^p$  reflexive, 6 pt). Prove that for every  $1 < p < \infty$  the space  $\ell^p$  is reflexive.

**Hint:** Relate the canonical map  $\iota_{\ell^p} : \ell^p \rightarrow (\ell^p)''$  to the map

$$\Phi_p : \ell^{p'} \rightarrow (\ell^p)'$$

**Problem 8** (spectrum of multiplication operator on  $\ell^p$ , 6 pt). Let  $p \in [1, \infty]$  and  $y \in \ell^\infty = \ell^\infty(\mathbb{N})$ . We define

$$M_y : \ell^p \rightarrow \ell^p, \quad M_y x := yx = (y^i x^i)_{i \in \mathbb{N}}.$$

Prove that

$$\begin{aligned} \sigma_{\text{pt}}(M_y) &= \text{im}(y) = \{y^i \mid i \in \mathbb{N}\}, \\ \sigma(M_y) &= \overline{\text{im}(y)}, \end{aligned}$$

where  $\sigma_{\text{pt}}$  denotes the point spectrum (= set of eigenvalues) and  $\sigma$  denotes the spectrum.

**Problem 9** (dual space of inner product space, 16 pt). Let  $(X, \langle \cdot, \cdot \rangle)$  be a real inner product space. Prove that there exists a linear isometry from  $X$  to its dual space  $X'$ , whose image is dense.

**Remark:** In this exercise you may use any exercise from the assignments (and any result from the lecture and the book by Rynne and Youngson).