Measure and Integration: Hertentamen 2013-14

(1) Consider the measure space $([0,1]\mathcal{B}([0,1]),\lambda)$, where $\mathcal{B}([0,1])$ is the restriction of the Borel σ-algebra to $[0,1]$, and λ is the restriction of Lebesgue measure to $[0,1]$. Let $E_1,\cdots,E_m$ be a collection of Borel measurable subsets of $[0,1]$ such that every element $x\in[0,1]$ belongs to at least n sets in the collection $\{E_j\}_{j=1}^m$, where $n\leq m$. Show that there exists a $j\in\{1,\cdots,m\}$ such that $\lambda(E_j)\geq\dfrac{n}{m}$. (1.5 pt)

(2) Let $(X,\mathcal{F},\mu)$ be a measure space, and $1<p,q<\infty$ conjugate numbers, i.e. $1/p+1/q=1$. Show that if $f\in\mathcal{L}^p(\mu)$, then there exists $g\in\mathcal{L}^q(\mu)$ such that $||g||_q=1$ and $\int fg\,d\mu=||f||_p$. (1.5 pt)

(3) Consider the measure space $(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$, where $\mathcal{B}(\mathbb{R}))$ is the Borel σ-algebra and λ is Lebesgue measure. Let $f\in\mathcal{L}^1(\lambda)$ and define for $h>0$, the function $f_h(x)=\dfrac{1}{h}\displaystyle\int_{[x,x+h]}f(t)\,d\lambda(t)$.

(a) Show that f_h is Borel measurable for all $h>0$. (1 pt)
(b) Show that $f_h\in\mathcal{L}^1(\lambda)$ and $||f_h||_1\leq||f||_1$. (1 pt)

(4) Consider the measure space $(\mathbb{R},\mathcal{B}(\mathbb{R}),\lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure.

(a) Show that for any $u\in\mathcal{L}^1(\lambda)$, one has $u\mathbf{1}_{[n-1/n,n+1/n]}\xrightarrow{\lambda}0$. (1 pt)
(b) Show that for any $u\in\mathcal{L}^1(\lambda)$, the sequence $(|u|\mathbf{1}_{[n-1/n,n+1/n]})$ is uniform integrable. (1 pt)
(c) Show that for any $u\in\mathcal{L}^1(\lambda)$ one has, $\lim_{n\to\infty}\int_{[n-1/n,n+1/n]}u\,d\lambda=0$. (1 pt)

(5) Let $(X,\mathcal{A},\mu)$ be a measure space and $f\in\mathcal{L}^1(\mu)$. Define $A_n=\{x\in X:1/n\leq|f(x)|<n\}$, for $n\geq1$. Show that for every $\epsilon>0$, there exists a positive integer N, such that $\mu(A_N)<\infty$ and $\int_{A_N^c}|f|\,d\mu<\epsilon$. (2 pts)