Utrecht University
Mathematics
Infinitesimaalrekening C (WISB234)
Block 3, 2012

Test, April 16, 2012

JUSTIFY YOUR ANSWERS
Allowed: material handed out in class and *handwritten* notes (*your handwriting*)

NOTE:

- The test consists of five problems plus one bonus problem.
- The score is computed by adding all the credits up to a maximum of 100

Problem 1. A particle of mass m is travelling along the circular helix

$$\vec{c}\,(t) \;=\; \big(3\cos(2\pi t)\,,\,3\sin(2\pi t)\,,\,t\big) \quad t \geq 0\,.$$

(a) (5 pts.) Determine the equation of the line tangent to the trajectory at $t = 1/8$.

(b) (5 pts.) Compute the distance travelled by the particle up to $t = 4$.

(c) (5 pts.) The force acting on the particle is $\vec{F} = m\vec{c}\,''$ (Newton's law). Compute the line integral of this force (work) along the trajectory up to $t = 1$.

Answers:

(a) $L(s) = \vec{c}\,'(1/8)s + \vec{c}\,(1/8)$ *or*

$$\begin{aligned} x &= -3\pi\sqrt{2}\,s + 3\sqrt{2}/2 \\ y &= 3\pi\sqrt{2}\,s + 3\pi\sqrt{2}/2 \\ z &= s + (1/8) \end{aligned}$$

Elliminating s the following non-parametric equations are obtained:

$$\begin{aligned} x + y &= 3\sqrt{2} \\ x + 3\pi\sqrt{2}z &= (3\sqrt{2}/8)(\pi + 4)\,. \end{aligned}$$

(b) $\int_0^4 \|\,\vec{c}\,'\,(t)\|\,dt \;=\; \int_0^4 \sqrt{36\pi^2 + 1}\,dt \;=\; 4\sqrt{36\pi^2+1}.$

(c) The line integral is zero because $\vec{F} = m\,4\pi^2\big(-3\cos(2\pi t)\,,\,-3\sin(2\pi t)\,,\,0\big)$ *is orthogonal to* $\vec{c}\,' = \big(-6\pi\sin(2\pi t)\,,\,6\pi\cos(2\pi t)\,,\,1\big)$.

Problem 2. (15 pts.) Find

$$\int_C e^{x^2}dx + \ln(1+|y|)\,dy + e^z dz$$

where C is the image of the circular helix of the previous problem traversed once for $t = 0$ to $t = 1$.

Answer: The field $\vec{F} = \big(e^{x^2}, \ln(1+|y|), e^z\big)$ *is conservative because its curl is zero. It is possible, and easier, to consider instead of the helix the straight segment* $\vec{c}_1\,(t) = (3,0,t)$ *for* $t \in [0,1]$. *Therefore, the line integral is*

$$\int_0^1 \big(e^9, 0, e^t\big)\cdot\big(0,0,1\big)\,dt \;=\; e - 1\,.$$

Alternatively, $\vec{F} = \vec{\nabla} f$ *with*

$$f(x, y, z) = \int^{x} e^{\tilde{x}^2}\, d\tilde{x} + \int^{y} \ln(1 + |\tilde{y}|)\, d\tilde{y} + e^{z}$$

hence the line integral is $f(3, 0, 1) - f(3, 0, 0) = e - 1$.

Problem 3. Consider the paraboloid

$$x^2 + y^2 - 4z \; = \; 0 \, .$$

(a) (5 pts.) Determine a parametrization of this surface.

(b) (5 pts.) Prove that the vector $\vec{N} = (2x, 2y, -4)$ is perpendicular at the surface at the point (x, y, z).

(c) (5 pts.) Find the equation of the plane tangent to the paraboloid at the point $(1, 1, 1/2)$.

(d) (10 pts.) Find the area of the paraboloid between $z = 0$ and $z = 1$.

Answers:

(a) The natural parametrization is as a function: $z = (x^2 + y^2)/4$, $x, y \in \mathbb{R}$, *but there are many others.*

(b) The paraboloid is the level surface $f = 0$ *of the function* $f = x^2 + y^2 - 4z$. *Hence* $\vec{\nabla} f = \vec{N}$ *is orthogonal to it.*

(c) The equation is $\vec{N}(1, 1, 1/2) \cdot \big(x - 1, y - 1, z - 1/2\big) = 0$, *that is*

$$(2, 2, 4) \cdot \big(x - 1, y - 1, z - 1/2\big) \; = \; 0 \quad \Longrightarrow \quad x + y - 2z = 1 \, .$$

(d) 1st way: *As* $z = g(x, y)$, *the area is*

$$\int\!\!\int_D \sqrt{1 + \Big(\frac{\partial g}{\partial x}\Big)^2 + \Big(\frac{\partial g}{\partial y}\Big)^2}\, dx\, dy \; = \; \int\!\!\int_D \sqrt{1 + \frac{x^2 + y^2}{4}}\, dx\, dy$$

with $D = \big\{(x, y) : x^2 + y^2 \leq 4\big\}$. *Passing to polar coordinates this gives*

$$2\pi \int_0^2 \rho \sqrt{1 + \frac{\rho^2}{4}}\, d\rho \; = \; 2\pi\, \frac{4}{3} \Big[1 + \frac{\rho^2}{4}\Big]^{3/2}\Big|_0^2 \; = \; \frac{8\pi}{3} \big[2\sqrt{2} - 1\big] \, . \tag{1}$$

2nd way: *The surface is formed by revolving* $f(x) = x^2/4$, $0 \leq x \leq 2$, *around the* z *axis. Therefore, its area is*

$$2\pi \int_0^2 |x| \sqrt{1 + \big[f'(x)\big]^2}\, dx \; = \; 2\pi \int_0^2 x \sqrt{1 + \frac{x^2}{4}}\, dx \, ,$$

which coincides with (1).

Problem 4. Consider the field

$$\vec{F}(x, y, z) \; = \; \left(\frac{-y}{x^2 + y^2}, \frac{x}{x^2 + y^2}, 0\right) \, ,$$

and let

$$C_1 = \text{ circle } x^2 + y^2 = R^2 \, , \; z = 0$$

and

$$C_2 = \text{ circle } (x - 2R)^2 + (y - R)^2 = R^2 \, , \; z = 0 \, ,$$

$R > 0$, both traversed once in the counterclockwise direction.

(a) (5 pts.) Verify that $\vec{\nabla} \times \vec{F} = \vec{0}$ except in the z axis.

(b) (10 pts.) Show that $\int_{C_1} \vec{F} \cdot d\vec{s} = 2\pi$.

(c) (10 pts.) Show that $\int_{C_2} \vec{F} \cdot d\vec{s} = 0$.

Answers:

(a) Easy computation. At the z axis the field is not defined.

(b) The circle can be parametrized as $\vec{c} = (R\cos\theta, R\sin\theta, 0)$, $\theta \in [0, 2\pi]$. Hence the line integral is

$$\int_0^{2\pi} \left(\frac{-R\sin\theta}{R^2\cos^2\theta + R^2\sin^2\theta}, \frac{R\cos\theta}{R^2\cos^2\theta + R^2\sin^2\theta}, 0 \right) \cdot \big(-R\sin\theta, R\cos\theta, 0\big)\, d\theta \;=\; \int_0^{2\pi} d\theta \;=\; 2\pi$$

(c) Immediate from Stokes' theorem (or Green's).

Problem 5. (20 pts.) Calculate the outward flux of the field $\vec{F} = (3x, 3y, 3z)$ through the boundary of the region comprised between the unit sphere centered at the origin and the cylinder of unit radius and height two centered at the origin.

Answer: The divergence of $\vec{F}$ is constant and equal to 9, hence by Gauss' theorem the outward fluxis 9 times the volume of the region (=volume of cylinder - volume of sphere), or $9(2\pi - 4\pi/3) = 6\pi$.

Bonus problem

Bonus Let f and g be differentiable scalar functions on $\mathbb{R}^3$.

(a) (2 pts.) Prove that $\vec{\nabla} \cdot (f\, \vec{\nabla} g) = \vec{\nabla} f \cdot \vec{\nabla} g + f\, \nabla^2 g$.

(b) (5 pts.) Prove the first Green identity

$$\iint_{\partial W} f\, \vec{\nabla} g\, d\vec{S} = \iiint_W \big(f\, \nabla^2 g + \vec{\nabla} f \cdot \vec{\nabla} g\big)\, dV\ .$$

(c) (5 pts.) Prove the second Green identity

$$\iint_{\partial W} \big(f\, \vec{\nabla} g - g\, \vec{\nabla} f\big)\, d\vec{S} = \iiint_W \big(f\, \nabla^2 g - g\, \nabla^2 f\big)\, dV\ .$$

Answers:

(a) Simple computation

(b) By Gauss, the left hand-side is equal to $\iiint_W \vec{\nabla} \cdot (f\, \vec{\nabla} g)\, dV$, which is equal to the right-hand side due to the identity in (a).

(c) Substract from the identity in (b) the same identity with f and g interchanged.