Test, May 30, 2012

JUSTIFY YOUR ANSWERS
Allowed: material handed out in class and *handwritten* notes (*your handwriting*)

Problem 1. (10 pts.) A wire has the form of the parabola $y = x^2$, with $-1 \le x \le 1$. The density at each point (x, y) of the wire is equal to $|x|$. Calculate the total mass of the wire (remember that the mass is the path integral of the density).

Answer: The most natural parametrization is $\vec{c}(x) = (x, x^2)$, $x \in \mathbb{R}$. *Hence* $\|\vec{c}'(x)\| = \sqrt{1+4x^2}$ *and*

$$\text{Mass} = \int_{-1}^{1} |x| \sqrt{1+4x^2}\, dx = 2\int_0^1 x\sqrt{1+4x^2}\, dx = \frac{1}{6}\big(1+4x^2\big)^{3/2}\Big|_0^1 = \frac{1}{6}\left[5\sqrt{5}-1\right].$$

Problem 2. (10 pts.) Prove that the field $\vec{F} = (-z\sin x\cos y\,, -z\cos x\sin y\,, \cos x\, cosy)$ is conservative and find f such that $\vec{F} = \vec{\nabla} f$.

Answer: It is easy to verify that $\vec{\nabla} \times \vec{F} = \vec{0}$. *Integrating or guessing one obtains* $\vec{F} = \vec{\nabla} f$ *with* $f = z\cos x\sin y$.

Problem 3. Consider the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

(a) (5 pts.) Find a parametrization of the curve obtained when traversing the ellipse once in a counter-clockwise direction, starting from the point $(a, 0)$.

(b) (5 pts.) Show that the line tangent to the ellipse at $(a/\sqrt{2}, -b/\sqrt{2})$ intersects the x-axis at $x = 2a/\sqrt{2}$.

(c) (10 pts.) Find the line integral of $\vec{F}(x, y) = (-y, x)$ along the curve.

(d) (10 pts.) Use Green's theorem to compute the area inside the ellipse.

Answers:

(a) $\vec{c}(\theta) = \big(a\cos\theta\ b\sin\theta\big)$, $\theta \in [0, 2\pi]$.

(b) First, observe that the condition $\big(a/\sqrt{2}, -b/\sqrt{2}\big) = \vec{c}\,(\theta)$ *implies* $\theta = 2\pi - (\pi/4)$. *Second,* $\vec{c}'(\theta) = \big(-a\sin\theta\,, b\cos\theta\big)$, *hence the line is in the direction of* $\vec{c}'(7\pi/4) = \big(a/\sqrt{2}\,, b/\sqrt{2}\big)$. *The parametric equation of the tangent is, therefore,*

$$\vec{\ell}(t) = \Big(\frac{a}{\sqrt{2}}, -\frac{b}{\sqrt{2}}\Big) + t\Big(\frac{a}{\sqrt{2}}, \frac{b}{\sqrt{2}}\Big) = \Big(\frac{a(1+t)}{\sqrt{2}}, \frac{b(t-1)}{\sqrt{2}}\Big).$$

The y-coordinate is zero if and only if $t = 1$, in which case the x-coordinate takes the value $2a/\sqrt{2}$.

(c)

$$\int_0^{2\pi} \big(-b\sin\theta\,, a\cos\theta\big)\cdot\big(-a\sin\theta\,, b\cos\theta\big)\, d\theta = ab\int_0^{2\pi}\left[\cos^2\theta + \sin^2\theta\right] d\theta = 2\pi ab.$$

(d)

$$\text{Area} = \frac{1}{2}\int_{\vec{c}} xdy - ydx = \frac{1}{2}\int_0^{2\pi} \big(-b\sin\theta\,, a\cos\theta\big)\cdot\big(-a\sin\theta\,, b\cos\theta\big)\, d\theta = \pi ab.$$

Problem 4. Consider the upper cone

$$\begin{aligned} x^2 + y^2 - z^2 &= 0 \\ z &\geq 0 , \end{aligned}$$

the field

$$\vec{F}(x, y, z) = (-x, -y, 2z)$$

and the sphere

$$x^2 + y^2 + z^2 = 2\sqrt{2}\, z .$$

(a) (5 pts.) Draw the sphere and the cone on the same set of coordinate axis.

(b) (5 pts.) Determine a parametrization of the cone.

(c) (5 pts.) For which point does not the cone have a tangent plane? Justify your assertion.

(d) (10 pts.) Determine the flux of the field $\vec{F}$ towards the outside of the cone, for $0 \leq z \leq 1$.

(e) (10 pts.) Determine the outward flux of $\vec{F}$ out of the closed surface formed by the portion of the cone inside the sphere capped by the portion of the sphere inside the cone.

Answers:

(a) $\vec{\Phi}(\rho, \theta) = \big(\rho\cos\theta\,,\ \rho\sin\theta\,,\ \rho\big)$, $\theta \in [0, 2\pi]$ *and* $\rho \geq 0$.

(b) The outward normal is

$$\frac{\partial \vec{\theta}}{\partial \theta} \times \frac{\partial \vec{\theta}}{\partial \rho} = \big(\rho\cos\theta\,,\ \rho\sin\theta\,,\ -\rho\big) .$$

The only point without tangent plane is the only point where the normal becomes the zero vector, namely $\rho = 0$ *or* $(x, y, z) = (0, 0, 0)$.

(c)

$$\begin{aligned} \int_S \vec{F} d\vec{S} &= \int_0^{2\pi} d\theta \int_0^1 d\rho \,\big(-\rho\cos\theta\,,\ -\rho\sin\theta\,,\ 2\rho\big)\cdot\big(\rho\cos\theta\,,\ \rho\sin\theta\,,\ -\rho\big) \\ &= 2\pi \int_0^1 d\rho\,(-3\rho^2) \\ &= -2\pi . \end{aligned}$$

(d) $\vec{\nabla} \cdot \vec{F} = 0$, *hence by Gauss the outward flux is zero.*

Problem 5. Let $\vec{v} = (v_1, v_2, v_3)$ be a constant vector and denote $\vec{r} = (x, y, z)$.

(a) (5 pts.) Compute $\vec{\nabla} \times (\vec{v} \times \vec{r})$.

(b) (10 pts.) Show that for every simple surface S,

$$\frac{1}{2} \int_{\partial S} (\vec{v} \times \vec{r}) \cdot d\vec{s} = \int\!\!\int_S \vec{v} \cdot d\vec{S} .$$

Answers:

(a) $\vec{\nabla} \times (\vec{v} \times \vec{r}) = 2\,\vec{v}$.

(b) Use Stokes.