

## Inleiding Topologie, Exam A (April 17, 2013))

**Note 1:** The mark for this exam is the minimum between 10 and the number of points that you score (in total, there are 13 points in the game!).

**Note 2:** please MOTIVATE ALL YOUR ANSWERS (e.g., in Exercise 2, do not just give the example, but also explain/prove why it has the required properties).

**Exercise 1.** (1 pt) Let  $X$  and  $Y$  be two topological spaces. For  $A \subset X$ ,  $B \subset Y$ , we consider  $A \times B$  as a subset of  $X \times Y$ . Show that:

$$\text{Int}(A \times B) = \text{Int}(A) \times \text{Int}(B)$$

(the interior of  $A \times B$  inside  $X \times Y$  (with respect to the product topology) = the product of the interior of  $A$  in  $X$  with the interior of  $B$  in  $Y$ ).

**Exercise 2.** (1 pt) Give an example of a connected, bounded, open subset of  $\mathbb{R}^2$  which cannot be written as a finite union of balls (here we use the Euclidean metric and topology on  $\mathbb{R}^2$ ).

**Exercise 3.** Let  $X = (-1, \infty)$ .

(i) (1 pt) Find all the numbers  $a, b \in \mathbb{R}$  with the property that

$$n \cdot t = \phi_n(t) = 2^n t + a^n + b$$

defines an action of the group  $(\mathbb{Z}, +)$  on  $X$ .

(ii) (1 pt) For the  $a$  and  $b$  that you found, show that the resulting quotient space  $X/\mathbb{Z}$  is homeomorphic to  $S^1$ .

**Exercise 4.** On  $X = \mathbb{Z}$  we consider the family  $\mathcal{B}$  of subsets of  $X$  consisting of the empty set and the subsets of type

$$N_{a,b} := a + b\mathbb{Z} = \{a + bn : n \in \mathbb{Z}\} \subset \mathbb{Z},$$

with  $a, b \in \mathbb{Z}$ ,  $b > 0$ .

(i) (0.25 pts) Show that, for  $a, a', b, b_1, b_2 \in \mathbb{Z}$  with  $b, b_1, b_2 > 0$ :

$$N_{a,1} = N_{a,-1} = \mathbb{Z},$$

$$N_{a,b_1 b_2} \subset N_{a,b_1} \cap N_{a,b_2}$$

and one has the following equivalences:

$$a' \in N_{a,b} \iff a \in N_{a',b} \iff N_{a,b} = N_{a',b}.$$

- (ii) (0.5 pts) Show that  $\mathcal{B}$  is not a topology on  $X$ .
- (iii) (0.5 pts) Show that  $\mathcal{B}$  is a topology basis on  $X$ . Let  $\mathcal{T}$  be the induced topology.
- (v) (1 pt) Compute the interior and the closure of  $A := \{-1, 1\}$  in  $(X, \mathcal{T})$ .
- (iv) (0.5 pts) Show that  $(X, \mathcal{T})$  is Hausdorff.
- (vi) (0.25 pts) Show that, for any  $b \in \mathbb{Z}$ ,  $b > 0$ ,  $\mathbb{Z}$  can be written as a union of  $b$  nonempty subsets that belong to  $\mathcal{B}$ , each two of them being disjoint.
- (vii) (0.5 pts) Show that any subset of type  $N_{a,b}$  is both open and closed in  $(X, \mathcal{T})$ .
- (viii) (1 pt) Show that

$$\mathbb{Z} \setminus \{-1, 1\} = \bigcup_{p\text{-prime number}} N_{0,p}$$

and then, using (vii) and (v), deduce that the set of prime numbers is infinite.

**Exercise 5.** Consider the 3-sphere  $S^3$  viewed as a subspace of  $\mathbb{C}^2$ :

$$S^3 = \{(u, v) : u, v \in \mathbb{C}, |u|^2 + |v|^2 = 1\}.$$

Inside the sphere we consider

$$A := \{(u, v) \in S^3 : |v| = \frac{\sqrt{2}}{2}\}.$$

- (i) (1 pt) Show that  $S^3 \setminus A$  has two connected components.
- (ii) (0.5 pts) Show that the two connected components, denoted  $X_1$  and  $X_2$ , satisfy:

$$\overline{X_1} \cap \overline{X_2} = \partial(X_1) = \partial(X_2) = A.$$

(where the closures and boundaries are inside the space  $S^3$ ).

- (iii) (1 pt) Consider the unit circle and the closed unit disk

$$S^1 = \{(\alpha, \beta) \in \mathbb{R}^2 : \alpha^2 + \beta^2 = 1\}, \quad D^2 = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}.$$

By a solid torus we mean any space homeomorphic to  $S^1 \times D^2$ . Show that

$$f : S^1 \times D^2 \rightarrow \mathbb{R}^3, \quad f((\alpha, \beta), (x, y)) = ((2-x)\alpha, (2-x)\beta, y)$$

is an embedding and indicate on a picture what the image of  $f$  is (... motivating the name "solid torus").

- (iv) (1 pt) Show that  $\overline{X_i}$  is a solid torus for  $i \in \{1, 2\}$ .
- (v) (1 pt) Deduce that the 3-sphere can be obtained from two disjoint copies of  $S^1 \times D^2$  (i.e. two solid tori) by gluing any point  $(z_1, z_2) \in S^1 \times S^1$  in the boundary of the first copy with the point  $(z_2, z_1)$  in the boundary of the second.