

## Inleiding Topologie, Exam B (June 26, 2013)

### Notes:

- English or Dutch (or both)- doesn't matter, but please write clearly (thanks!).
- Please motivate your answers. For instance, in exercise 4, part (ii), do not forget, after you write the function  $f$ , to check that  $f$  is well defined, continuous and that it has the desired property.
- In this exam, the sub-points of any exercise do not fully depend on each other. For instance, in the last exercise, you may do (v) without doing (iv) (... but using it).

**Exercise 1.** Prove that there is no continuous injective map  $f : S^2 \longrightarrow S^1$ . (1p)  
(warning: there are injective maps from  $S^2$  to  $S^1$ !).

**Exercise 2.** Let  $X = (0, \infty)$  and consider the open cover of  $X$

$$\mathcal{U} = \{U_1, U_2, U_3, \dots\} \quad \text{with } U_n = (0, n).$$

- (i) Show that  $\mathcal{U}$  does not admit a subcover which is locally finite. (0.5p)
- (ii) Describe a locally finite refinement of  $\mathcal{U}$ . (0.5p)

**Exercise 3.** Consider

$$X = \{(u, v, w) \in \mathbb{R}^3 : u^2 + v^2 + w^2 = 1\} \setminus \{(0, 0, 1), (0, 0, -1)\},$$

$$Y = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 2\sqrt{x^2 + y^2}\} \setminus \{(0, 0, 0)\}.$$

$$\pi : X \longrightarrow \mathbb{R}^3, \quad \pi(u, v, w) = (2u\sqrt{u^2 + v^2}, 2v\sqrt{u^2 + v^2}, 2w\sqrt{u^2 + v^2}).$$

- (i) Compute the closure  $\bar{Y}$  of  $Y$  in  $\mathbb{R}^3$  and prove that it is compact. (1p)
- (ii) Prove that the one-point compactification of  $X$  is homeomorphic to  $\bar{Y}$ . (1p)
- (iii) Draw a picture of  $Y$  and explain the map  $\pi$  on the picture. (1p)

**Exercise 4.** Let  $X$  be a Hausdorff, locally compact, 2<sup>nd</sup> countable topological space and assume that  $U$  is an open in  $X \times \mathbb{R}$  containing  $X \times \{0\}$ :

$$X \times \{0\} \subset U \subset X \times \mathbb{R}.$$

The aim of this exercise is to prove that there exists a continuous function  $f : X \longrightarrow (0, \infty)$  such that  $U$  contains

$$U_f := \{(x, t) \in X \times \mathbb{R} : |t| < f(x)\}.$$

Consider

$$r : X \longrightarrow \mathbb{R}, \quad r(x) = \sup\{r \in (0, 1] : \{x\} \times (-r, r) \subset U\}.$$

- (i) Show that one can find an open cover  $\{V_i : i \in I\}$  of  $X$  and a family  $\{r_i : i \in I\}$  of strictly positive real numbers (for some indexing set  $I$ ) such that

$$r(y) > r_i \quad \forall y \in V_i, \forall i \in I. \quad (0.5p)$$

(note: depending on the argument that you find,  $I$  that you construct may be countable, but it may also be “very large”- e.g. “as large as  $X$ ”).

- (ii) For  $\{V_i : i \in I\}$ ,  $\{r_i : i \in I\}$  as above, use a partition of unity argument to build a continuous function  $f : X \rightarrow (0, \infty)$  such that  $U_f \subset U$ . (1.5p)
- (iii) Deduce that if  $X$  is actually compact, then  $f$  may be chosen to be constant. (0.5p)

**Exercise 5.** In this exercise we work over  $\mathbb{R}$ . Let  $X$  be a compact, Hausdorff topological space,  $C(X)$  the space of real-valued continuous functions on  $X$  and let

$$\mathcal{A} \subset C(X).$$

be a point-separating subalgebra. The aim of this exercise is to show that the spectrum  $X_{\mathcal{A}}$  is homeomorphic to  $X$ . The homeomorphism will be provided by the map:

$$F : X \rightarrow X_{\mathcal{A}}, \quad F(x) = \chi_x|_{\mathcal{A}}$$

(the restriction of  $\chi_x : C(X) \rightarrow \mathbb{R}$  to  $\mathcal{A}$ , where we recall that  $\chi_x$  sends  $f$  to  $f(x)$ ).

- (i) Show that  $F$  is continuous. (0.5p)
- (ii) Show that  $F$  is injective. (0.5p)

The next five steps are to prove that  $F$  is surjective. Let  $\chi \in X_{\mathcal{A}}$  be a character of  $\mathcal{A}$ .

- (iii) Show that, if  $f, g \in \mathcal{A}$  and  $f \geq g$ , then  $\chi(f) \geq \chi(g)$ . (0.5p)

(Hint: recall that, in the proof of the Stone-Weierstrass theorem we showed that, for  $f \in \mathcal{A}$  with  $f \geq 0$ , one has  $\sqrt{f} \in \overline{\mathcal{A}}$ ).

- (iv) Show that for all  $f \in \mathcal{A}$  one has  $|\chi(f)| \leq \|f\|_{\text{sup}}$ . (0.5p)
- (v) Deduce that for any sequence  $(f_n)_{n \geq 1}$  of elements in  $\mathcal{A}$ , convergent to some  $f \in C(X)$ , the sequence  $(\chi(f_n))_{n \geq 1}$  is convergent. (0.5p)
- (vi) Deduce that there exists an extension of  $\chi : \mathcal{A} \rightarrow \mathbb{R}$  to a continuous map

$$\tilde{\chi} : C(X) \rightarrow \mathbb{R}. \quad (0.5p)$$

- (vii) And then show that  $\tilde{\chi}$  is a character. (0.5p)

Finally:

- (viii) Conclude that  $F$  is a homeomorphism. (0.5p)