Inleiding Topologie, Exam B (June 29, 2011)

Exercise 1. Consider

$$X_1 = \{(x, y, z) \in \mathbb{R}^3 : (z = 0) \text{ or } (x = y = 0, z \geq 0)\},$$
$$X_2 = \{(x, y, z) \in \mathbb{R}^3 : (z = 0) \text{ or } (x = 0, y^2 + z^2 = 1, z \geq 0)\}.$$
$$X_3 = \{(x, y, z) \in \mathbb{R}^3 : (x^2 + y^2 + z^2 = 1) \text{ or } (y = 0, z = 0, \frac{1}{2} < |x| < 1)\},$$

(i) Show that X_1, X_2, X_3 are locally compact (hint: try to use the basic properties of locally compact spaces; alternatively, you can try to find direct arguments on the pictures). (*0.5 p*)

(ii) Show that the one-point compactifications of X_1, X_2 and X_3 are homeomorphic to each other. (*1 p*)

Exercise 2. Given a polynomial $p \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, we denote by $\mathcal{R}_p$ the set of reminders modulo p. In other words,

$$\mathcal{R}_p = \mathbb{R}[X_0, X_1, \ldots, X_n]/R_p,$$

where R_p is the equivalence relation on $\mathbb{R}[X_0, X_1, \ldots, X_n]$ given by

$$R_p = \{(q_1, q_2) : \exists\, q \in \mathbb{R}[X_0, X_1, \ldots, X_n] \text{ such that } q_1 - q_2 = pq\}.$$

We also denote by $\pi_p : \mathbb{R}[X_0, X_1, \ldots, X_n] \longrightarrow \mathcal{R}_p$ the resulting quotient map. Show that:

(i) There is a unique algebra structure on $\mathcal{R}_p$ (i.e. unique operations $+$, $\cdot$ and multiplications by scalars, defined on $\mathcal{R}_p$) with the property that π_p is an morphism of algebras, i.e.

$$\pi_p(q_1 + q_2) = \pi_p(q_1) + \pi_p(q_2),\ \pi_p(q_1 \cdot q_2) = \pi_p(q_1) \cdot \pi_p(q_2),\ \lambda\pi_p(q) = \pi_p(\lambda q)$$

for all $q_1, q_2 \in \mathbb{R}[X_0, X_1, \ldots, X_n]$, $\lambda \in \mathbb{R}$. (*0.5 p*)

(ii) For $p = x_0^2 + \ldots + x_n^2$, the spectrum of $\mathcal{R}_p$ has only one point. (*1 p*)

(iii) For $p = x_0^2 + \ldots + x_n^2 - 1$, the spectrum of $\mathcal{R}_p$ is homeomorphic to S^n (*1 p*) .

(iv) What is the spectrum for $p = x_0 x_1 \ldots x_n$? (*0.5 p*)

Exercise 3. Let X be the space of continuous maps $f : [0, 1] \longrightarrow [0, 1]$ with the property that $f(0) = f(1)$. We endow it with the sup-metric d_{sup} and the induced topology (recall that $d_{\text{sup}}(f, g) = sup\{|f(t) - g(t)| : t \in [0, 1]\}$). Prove that:

(i) X is bounded and complete. (*1 p*)

(ii) X is not compact. (*0.5 p*)

Exercise 4. Show that:

(i) The product of two sequentially compact spaces is sequentially compact. (*1 p*)

(ii) Deduce that the product of two compact metric space is a compact space. (*1 p*)

Exercise 5. Show that the family of open intervals

$$\mathcal{U} := \{(q, q+1) : q \in \mathbb{R}\}$$

forms an open cover of $\mathbb{R}$ which admits no finite sub-cover, but which admits a locally finite sub-cover. (*1.5 p*)

Exercise 6. Prove that there is no continuous injective map $f : S^1 \vee S^1 \longrightarrow S^1$, where $S^1 \vee S^1$ is a bouquet of two circles (two copies of S^1, tangent to each other). (*1.5 p*)

Note: The mark for this exam is the minimum between 10 and the number of points that you score (in total, there are 11 points in the game!).