# Exam Inleiding Topologie, WISB243 2018-01-29, 13:30 – 16:30

- Write your **name** on every sheet, and on the first sheet your **student number**, **group** (1: Lauran, 2: Luca) and the total **number of sheets** handed in.
- You may use the lecture notes, the extra notes and personal notes, but no worked exercises.
- Do not just give answers, but also justify them with complete arguments. If you use results from the lecture notes, always **refer to them by number**, and show that their hypotheses are fulfilled in the situation at hand.
- **N.B.** If you fail to solve an item within an exercise, do **continue**; you may then use the information stated earlier.
- The weights by which exercises and their items count are indicated in the margin. The highest possible total score is 55. The final grade will be obtained from your total score through division by 5.
- You are free to write the solutions either in English, or in Dutch.

*Succes !*

15 pt total **Exercise 1.** On $\mathbb{R}$ we consider the collection $\mathscr{B} \subset \mathscr{P}(\mathbb{R})$ consisting of $\mathbb{R}$ and all intervals of the form $[m,a)$, with $m \in \mathbb{Z}$ and $a \in \mathbb{R}$ (for convenience we agree that $[m,a) = \emptyset$ for $m \geq a$).

3 pt (a) Show that $\mathscr{B}$ is a topology basis, but not a topology.

Let $\mathscr{T}$ be the topology on $\mathbb{R}$ generated by $\mathscr{B}$.

2 pt (b) Is $\mathscr{T}$ Hausdorff?

2 pt (c) Is $\mathscr{T}$ second countable?

5 pt (d) Determine the interior and the closure of the set $A := [-\frac{1}{2}, \frac{1}{2}]$ with respect to $\mathscr{T}$.

3 pt (e) Determine for which $r > 0$ the subset $[0,r]$ is connected for the subspace topology (induced by $\mathscr{T}$). Prove the validity of your answer.

10 pt total **Exercise 2.** Let $X$ and $Y$ be non-empty topological spaces. We equip $X \times Y$ with the product topology.

5 pt (a) Show: if $X$ and $Y$ are Hausdorff spaces, then $X \times Y$ is a Hausdorff space.

5 pt (b) Show that the converse is also true: if $X \times Y$ is Hausdorff, then both $X$ and $Y$ are Hausdorff.

11 pt total **Exercise 3.** We consider a second countable locally compact Hausdorff space $X$. A function $\varphi : X \to [0,\infty)$ is said to be locally bounded if and only if for every $a \in X$ there exists a neighborhood $V$ of $a$ and a constant $M > 0$ such that $\varphi \leq M$ on $V$.

Let $f : X \to [0,\infty)$ be a function. Prove that the following assertions are equivalent.

(1) The function $f$ is locally bounded.

(2) There exists a continuous function $g : X \to [0,\infty)$ such that $f \leq g$ on $X$.

4 pt for the proof that '(2) $\Rightarrow$ (1)'.

7 pt for the proof that '(1) $\Rightarrow$ (2)'.

19 pt total **Exercise 4.** Let $S^1$ be the unit circle in $\mathbb{R}^2$. Let $\Gamma$ be the group $\{1,g\}$ of two elements, with group law given by $11 = g^2 = 1$ and $1g = g1 = g$.

We consider two actions $\alpha, \beta$ of $\Gamma$ by homeomorpisms on $S^1$. These actions are given by $\alpha_1 = \beta_1 = \mathrm{id}_{S^1}$ and

$$\alpha_g(x) = (-x_1, -x_2), \quad \beta_g(x) = (x_1, -x_2), \qquad (x = (x_1, x_2) \in S^1).$$

In the following you may use without proof that $\alpha$ and $\beta$ are indeed actions by homeomorphisms. For $\gamma \in \Gamma$ we define $\rho_\gamma : S^1 \times S^1 \to S^1 \times S^1$ by

$$\rho_\gamma((x,y)) = (\alpha_\gamma(x), \beta_\gamma(y)), \quad ((x,y) \in S^1 \times S^1).$$

2 pt (a) Show that $\rho$ is an action of $\Gamma$ on $S^1 \times S^1$ by homeomorphisms.

1 pt (b) For each point $p = (x,y)$ of the torus $S^1 \times S^1$, determine the orbit $\Gamma p$.

In the following, $q$ denotes the natural quotient map $S^1 \times S^1 \to S^1 \times S^1/\Gamma$ associated with the action $\rho$. We define the map $f : [0,1] \times S^1 \to S^1 \times S^1$ by $f(s,y) = (\cos \pi s, \sin \pi s, y)$.

3 pt (c) Show that $f$ is a topological embedding.

3 pt (d) Show that $F := q \circ f$ is surjective from $[0,1] \times S^1$ onto the quotient $S^1 \times S^1/\Gamma$.

Let $\sim$ be the equivalence relation on $[0,1] \times S^1$ determined by $z \sim z' \iff F(z) = F(z')$.

4 pt (e) Prove that $S^1 \times S^1/\Gamma$ is homeomorphic to $[0,1] \times S^1/\sim$.

3 pt (f) Show that $F$ is bijective from $[0,1) \times S^1$ onto $S^1 \times S^1/\Gamma$.

3 pt (g) Calculate the equivalence classes of $\sim$ in $[0,1] \times S^1$. Use a picture to indicate why $[0,1] \times S^1/\sim$ is homeomorphic to the Klein bottle.