Final exam, Mathematical Modelling (WISB357)

Wednesday, 1 February 2017, 9.00-12.00, BBG 161

- Write your name on each page you turn in, and additionally, on the first page, write your student number and the *total number of pages submitted.*
- For each question, motivation your answer.
- You may make use of results from previous subproblems, even if you have been unable to prove them.
- For this midterm exam you are allowed to bring an A4 with notes on one side. You may not consult solutions to the problems, nor use a graphical calculator or smart phone.

Problem 1. This problem concerns the "reaction-diffusion" equation

$$u_t = Du_{xx} - cu, \quad \text{for} \quad \begin{cases} -\infty < x < \infty, \\ 0 < t, \end{cases}$$

with the initial condition

$$u(x,0) = f(x).$$

Assume c and D are positive constants.

(a) Using the Fourier Transform, find the solution of the above problem.

(b) Show that the problem can also be solved by applying the transformation $u = ve^{at}$, for a carefully chosen constant a, followed by using known solution of the diffusion equation.

Problem 2. Suppose "traffic" is governed by the Burgers equation

$$\rho_t + \rho\rho_x = 0,$$

with initial condition

$$\rho(x,0) = \begin{cases} 0, & x \le -1, \\ \frac{1}{2}(1+x), & -1 < x < 1, \\ 1, & 1 \le x. \end{cases}$$

(a) Sketch the characteristics in the (x,t)-plane.

(b) Find the solution, $\rho(x,t)$, using the method of characteristics.

(c) Find the points in the (x,t)-plane where $\rho = 1/3$.

(d) Show that $v = \frac{1}{2}\rho$. Determine the flux J.

<u>Problem 3</u>. A linearly elastic bar is made of two different materials, and before being stretched it occupies the interval $0 \le A \le \ell_0$. Also, before being stretched, for $0 \le A < A_0$, the modulus and density are $E = E_L$ and $R = R_L$, while for $A_0 < A < \ell_0$ they are $E = E_R$ and $R = R_R$. Both R_L and R_R are constants. (*Hint:* It is useful to define separate functions $U_L(A)$, $U_R(A)$, $T_L(A)$, $T_R(A)$, etc. on the left and right parts of the domain.)

(a) The requirements at the interface, where $A = A_0$, are that the displacement and stress are continuous. Express these requirements mathematically, using one-sided limits.

(b) Suppose the bar is stretched and the boundary conditions are $U(0,t) = 0$ and $U(\ell_0, t) = \ell - \ell_0$. Assume there are no body forces. Find the steady state solution for the density, displacement and stress.