
JUSTIFY YOUR ANSWERS

Allowed material: calculator, material handed out in class and *handwritten* notes (*your handwriting*). NO BOOK IS ALLOWED

NOTE: The test consists of six questions for a total of 12 points. The score is computed by adding all the credits up to a maximum of 10

Exercise 1. [Mortgage with adjustable interest] To buy a house you subscribe a loan for 200000 euros to be reimbursed monthly during 20 years. The contract determines that the bank has the right to adjust the interest every 5 years according to market values. At the moment the loan is signed the effective interest rate is 4% per year.

- (a) (0.3 pts.) Determine the amount of the monthly payment initially agreed.
- (b) At the end of the first 5-th year period, determine
 - i- (0.3 pts.) The part of the principal reimbursed so far.
 - ii- (0.3 pts.) The total amount payed so far in interest.
- (c) (0.6 pts.) The interest is adjusted only after 10 years, at which time it grows to 5% per year. Determine the new monthly payment.

Exercise 2. [Variance of increments of martingales] (1.5 pts.) Prove that for every martingale $(M_n)_{n \geq 0}$ and for every $k \geq 0$,

$$\text{Var}(M_{n+k} - M_n) = \text{Var}(M_{n+k}) - \text{Var}(M_n).$$

Exercise 3. [Asian option] Consider a stock with initial price S_0 following a binomial model with $u = 2$ and $d = 1/2$. That is, at the end of each period, the price can either double or be halved. Bank interest is 10% for each period. An investor needs the stock at the end of three periods and wishes to pay at most S_0 at that time.

- (a) The investor purchases an Asian call option with strike value S_0 , that is an option that can only be exercised at the end of the third period, with payoff

$$V_3 = \left| \frac{1}{4} \sum_{j=0}^3 S_j - S_0 \right|_+.$$

Compute:

- i- (0.5 pts.) The risk-neutral probability

-ii- (1 pt.) The initial price V_0 of the option.

- (b) At the end of the first period the investor finds that the price of the stock has been reduced to half its original value (that is, the market is in the situation “ T ”). The investor decides to sell his/hers call option and to purchase an American version of the option for the remaining two periods with strike value equal to the current stock value $S_0/2$. This is an option that can be exercised at the end of any of the three remaining periods, and offers intrinsic payoff

$$G_n = \frac{1}{n} \sum_{j=1}^n S_j(T, \dots) - S_1(T) \quad n = 1, 2, 3,$$

applying to the part of the market tree starting at $S_1(T)$. For this option:

- i- (1 pt.) Determine the net cost to the investor of the change of options, that is the price of the American option minus the selling price of the European option at time 1 (in situation “ T ”).
-ii- (1 pt.) Establish the optimal exercise time τ^* for the investor.
-iii- (1 pt.) Verify the validity of the formula

$$\text{Value of the American option} = \tilde{\mathbb{E}} \left[\mathbb{I}_{\{\tau^* \leq N\}} \frac{G_{\tau^*}}{(1+r)^{\tau^*}} \right].$$

- iv- (1 pt.) Show that the discounted values \bar{V}_n do *not* form a martingale, but the stopped discounted values $\bar{V}_n^{\tau^*}$ do.

Exercise 4. [True or false] Determine whether each of the following statements is true or false. If true provide a proof, if false provide a counterexample (you can copy examples from class notes or homework problems).

- (a) (0.4 pts.) Every martingale is a Markov process.
(b) (0.4 pts.) Every Markov process is a martingale.
(c) (0.4 pts.) For *all* American call options the optimal exercise time is at maturity or never.
(d) (0.4 pts.) The initial value of an American option is larger or equal than the initial value of its European version.

Exercise 5. [Exotic Black-Scholes-Merton option] (1.5 pts.) Consider a BSM market with initial stock price S_0 , (continuously compounded) risk-free yearly interest rate r , expected return μ per annum and volatility σ per annum. A call option is proposed with payoff

$$V(T) = \left| [\tilde{S}(T)]^2 - K \right|_+.$$

Compute its fair purchase value.

Exercise 6. [Convexity] (0.6 pts.) Let $f : \mathbb{R} \rightarrow \mathbb{R} \cup \{\infty\}$ be a convex function, that is, a function satisfying

$$f(\theta x + (1 - \theta)y) \leq \theta f(x) + (1 - \theta)f(y)$$

for all $x, y \in \mathbb{R}$ and all $0 \leq \theta \leq 1$. Show that for all $x, y, z \in \mathbb{R}$ with $x < y < z$,

$$\frac{f(y) - f(x)}{y - x} \leq \frac{f(z) - f(x)}{z - x}.$$