Utrecht University
Stochastic processes
WISB362
Spring 2017

Exam June 28, 2017

JUSTIFY YOUR ANSWERS!!

Please note:

- **Allowed: calculator, course-content material and notes *handwritten by you***
- **NO PHOTOCOPIED MATERIAL IS ALLOWED**
- **NO BOOK OR PRINTED MATERIAL IS ALLOWED**
- **If you use a result given as an exercise, you are expected to include (copy) its solution unless otherwise stated**

NOTE: The test consists of seven questions for a total of 10.5 points plus a bonus problem worth 1.5 pts. The score is computed by adding all the credits up to a maximum of 10

Exercise 1. [Composed randomness] (0.5 pts.) Consider IID random variables $(X_i)_{i\geq 1}$, with $X_i \sim \mathrm{Exp}(\lambda)$, and a further independent variable $N \sim \mathrm{Poisson}(\mu)$. Let $Y = \sum_{i=1}^{N} X_i$. Determine the moment-generating function $\Phi_Y(t) = E\big(\mathrm{e}^{tY}\big)$.

Exercise 2. [Generalization of a homework exercise] Consider a random variable Y with $E(Y) = \mu_Y$ and $\mathrm{Var}(Y) = \sigma_Y^2$. Let X be another random variable with a linear conditional mean and constant conditional variance:

$$\begin{aligned} E(X \mid Y) &= A + B\,Y \\ \mathrm{Var}(X \mid Y) &= \sigma_{X|Y}^2 \end{aligned}$$

with $A, B, \sigma_{X|Y}^2$ constant.

(a) (0.4 pts.) Compute $\mu_X := E(X)$ and deduce that

$$E(X \mid Y) \;=\; \mu_X + B\,(Y - \mu_Y)\;.$$

(b) (0.4 pts.) Compute $\sigma_X^2 := \mathrm{Var}(X)$ and deduce that

$$\sigma_{X|Y}^2 \;=\; \sigma_X^2 - B^2\sigma_Y^2\;.$$

(c) (0.4 pts.) Compute

$$\rho \;:=\; \frac{\mathrm{Cov}(X,Y)}{\sigma_X\sigma_Y} \;=\; \frac{E(XY) - E(X)E(Y)}{\sigma_X\,\sigma_Y}$$

and deduce that

$$\sigma_{X|Y}^2 \;=\; (1-\rho^2)\,\sigma_X^2\;.$$

Exercise 3. [Markov modelling] A monkey looks for food in three different regions, R_1, R_2 and R_3. The animal invest one hour looking for food in a given region and, after this, either remains a further hour in the region or jumps to one of the other regions. Each of these possibilities are found to be equiprobable. A group of biologists decides to capture the monkey to study its health situation. To this end they install a trap in region 3 that will capture the monkey immediately upon arrival there.

(a) (0.5 pts.) Model the resulting food search of the monkey via a Markov transition matrix.

(b) Assume that the monkey starts in the morning with a visit to region 1. Determine:

-i- (0.5 pts.) The law of T_3 = capture time.

-ii- (0.5 pts.) The mean and variance of the time biologists must wait to have the monkey.

Exercise 4. [Fun with exponentials] Let X_1, X_2 and X_3 be independent exponential random variables with respective rates λ_1, λ_2 and λ_3.

(a) (0.7 pts.) Compute $E\big(X_2^2 \mid X_1 < X_2 < X_3\big)$.

(b) (0.7 pts.) Consider now the order statistics $X_{(1)}, X_{(2)}, X_{(3)}$. Compute $E(X_{(2)})$.

Exercise 5. [Fun with Poisson processes] Let $N(t)$ be a Poisson process with rate λ. Compute

(a) (0.7 pts.) $E\big(N(2) \mid N(1) = 4\big)$.

(b) (0.7 pts.) $E\big(N(1) \mid N(2) = 4\big)$.

(c) (0.7 pts.) $\mathrm{Var}\big(N(2) \mid N(1) = 4\big)$.

Exercise 6. [Multiprocessors] A computer has k processors with identical independent exponential processing times with rate μ. Instructions are processed on a first-come first-serve basis as soon as a processor becomes free. Instructions arrive with independent exponentially distributed interarrival times with rate λ.

(a) An instruction arrives itself first in line with all processors busy. Denote W the waiting time of the instruction and T_P its processing time once accepted by a processor. Let $T = W + T_P$ be the total time elapsed between the arrival of the instruction and the completion of its processing. Determine:

-i- (0.7 pts.) The law of W.

-ii- (0.7 pts.) $E(T)$.

(b) (0.5 pts.) Write the number of instructions present as a birth-and-death chain, that is, determine the birth rates λ_n and death rates μ_n.

(c) Consider $k = 2$.

-i- (0.5 pts.) Determine the mean time needed for having three instructions present.

-ii- (0.6 pts.) Determine the limiting probabilities P_i, $i \geq 0$. Under which condition do these probabilities exist?

-iii- (0.6 pts.) Determine the values of λ/μ such that, in the long run the computer is idle 1/3 of the time.

Exercise 7. [Pure-death process (0.7 pts.) A *pure-death birth-and-death process* is a process with $\lambda_i = 0$ and, in consequence, $P_{ij}(t) = 0$ if $i < j$. Use Kolmogorov equations to determine $P_{ii}(t)$.

Bonus problem

Bonus 1. **P**rove that in an irreducible chain with finite alphabet all states are recurrent. The steps are the following.

(a) (0.5 pts.) Prove that for all $n \in \mathbb{N}$, $n \geq 1$, $\ell \leq n$ and $x, y \in \mathcal{A}$,

$$P\big(X_n = y, T_y = \ell \bigm| X_0 = x\big) \;=\; \mathbb{P}_{yy}^{n-\ell}\, P\big(T_y = \ell \bigm| X_0 = x\big)\,. \tag{1}$$

(b) (0.5 pts.) Show that, as a consequence, for all n and $x, y \in S$

$$P_{xy}^n \;=\; \sum_{\ell=1}^{n} \mathbb{P}_{yy}^{n-\ell}\, P\big(T_y = \ell \bigm| X_0 = x\big)\,. \tag{2}$$

(c) (0.2 pts.) Deduce that, if every state is transient,

$$\sum_n \mathbb{P}_{xy}^n \;<\; \infty$$

for every $x, y \in S$.

(d) (0.2 pts.) By summing over y obtain a contradiction with the assumed stochasticity of the matrix $\mathbb{P}$.

(e) (0.1 pt.) Conclude

Table 2.1

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters n, p, $0 \le p \le 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + (1-p))^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \le p \le 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

Table 2.2

Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters (n, λ), $\lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x} (\lambda x)^{n-1}}{(n-1)!}, & x \ge 0 \\ 0, & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^n$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \times \exp\{-(x-\mu)^2/2\sigma^2\}$, $-\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2