Utrecht University
Stochastic processes
WISB362
Spring 2017

Exam July 19, 2017

JUSTIFY YOUR ANSWERS!!

Please note:

- **Allowed: calculator, course-content material and notes *handwritten by you***
- **NO PHOTOCOPIED MATERIAL IS ALLOWED**
- **NO BOOK OR PRINTED MATERIAL IS ALLOWED**
- **If you use a result given as an exercise, you are expected to include (copy) its solution unless otherwise stated**

NOTE: The test consists of five questions for a total of 10.5 points plus a bonus problem worth 1.5 pts. The score is computed by adding all the credits up to a maximum of 10

Exercise 1. [Composed randomness] (0.7 pts.) Consider IID random variables $(X_i)_{i\geq 1}$, with $X_i \sim \text{Exp}(\lambda)$, and a further independent variable $N \sim \text{Poisson}(\mu)$. Let $Y = \sum_{i=1}^{N} X_i$. Determine the variance of Y.

Exercise 2. [Classes of states] Consider a Markov chain with alphabet $\{1, 2, 3, 4\}$ and transition matrix:

$$\begin{pmatrix} 1/4 & 3/4 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1/3 & 2/3 \end{pmatrix}$$

(a) (0.5 pts.) Determine the classes of states.

(b) Prove that

-i- (0.3 pts.) 4 is transient,

-ii- (0.3 pts.) 3 is absorbing,

-iii- (0.3 pts.) 1 and 2 are recurrent.

(c) (0.6 pts.) Determine *all* invariant measures (also called stationary measures).

Exercise 3. [Markov modelling I] A geyser emits boiling water following a random pattern. Each minute it is not emitting, the geyser has a 20% probability of emitting the following minute. Once emitting, the geyser has a 40% probability of continuing emitting during the next minute. The activity level corresponds, then, to a process $X_n = 1$ if the geyser is active at the n-th minute and 0 otherwise.

(a) (0.4 pts.) Find the transition matrix of the process X_n.

(b) (0.5 pts.) If the geyser is emitting now, find the probability that it will be emitting in four minutes.

(c) (0.7 pts.) Find the proportion of time the geyser emits.

(d) (0.7 pts.) The geyser is connected to a turbine that generates increasing power with the level of activity. There is no generation if the geyser does not emit for two consecutive minutes. If a non-active minute is followed by an emission, the turbine starts up and generates 1 Megawatt of power. If the geyser is active two consecutive minutes the turbine achieves maximum efficiency and generates 10 Megawatt. Finally, if an emission is followed by a non-active minute, the turbine manages to generate 3 Megawatt. Compute the mean power generated by the turbine.

Exercise 4. [Fun with Poisson processes] Let $\{N(t) : t \geq 0\}$ be a Poisson process with rate λ.

(a) For $t, s \geq 0$, determine:

- -i- (0.5 pts.) $P\big(N(t) = 1, N(t+s) = 2\big)$.
- -ii- (0.8 pts.) $E\big[N(t)\, N(t+s)\big]$.

(b) Let T_n denote the n-th inter-arrival time and S_n the arrival time of the n-th event. Find:

- -i- (0.5 pts.) $E\big[S_5 \,\big|\, S_2 = 3\big]$.
- -ii- (0.8 pts.) $E\big[T_3 \,\big|\, T_1 < T_2 < T_3\big]$.

(c) (0.8 pts.) Let S be a random variable independent of the Poisson process $\{N(t) : t \geq 0\}$. Show that

$$E\big(N(S^k)\big) \;=\; \lambda\, E(S^k)$$

for each $k \geq 1$.

Exercise 5. [Markov modelling II] A biological institute establishes a reserved sector for sick felines. Animals arrive at an exponential rate λ and they are so sick that they do not reproduce. Each animal dies at an independent exponential rate μ.

(a) (0.5 pts.) Set the population of the sector as a birth-and-death model by determining the birth and death rates λ_n and μ_n.

(b) (0.8 pts.) Determine the invariant measure of the model.

(c) (0.8 pts.) If $\lambda = \mu$, find the proportion of time in which there are three or more animals in the sector.

Bonus problem

Bonus. [Alternative definition of Poisson processes.] (1.5 pt.) The objective of this exercise is to prove part of the characterization of a Poisson process of rate λ as a counting process $N(t)$ of the form

$$N(t) \;=\; \max\{n : S_n \leq t\} \tag{1}$$

where

$$S_n \;=\; \sum_{i=1}^{n} T_i \tag{2}$$

for IID Exp(λ) random variables T_i, $i \geq 1$. Prove that (1) and (2) imply that

$$P\big(N(t) = n\big) \;=\; \mathrm{e}^{-\lambda t}\,\frac{(\lambda t)^n}{n!}\ .$$

[*Hint:* Note that $\big\{N(t) = n\big\} = \big\{S_n \leq t\,, T_{n+1} + S_n > t\big\}$.]

Table 2.1

Discrete probability distribution	Probability mass function, $p(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Binomial with parameters n, p, $0 \le p \le 1$	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0, 1, \ldots, n$	$(pe^t + (1-p))^n$	np	$np(1-p)$
Poisson with parameter $\lambda > 0$	$e^{-\lambda} \frac{\lambda^x}{x!}$, $x = 0, 1, 2, \ldots$	$\exp\{\lambda(e^t - 1)\}$	λ	λ
Geometric with parameter $0 \le p \le 1$	$p(1-p)^{x-1}$, $x = 1, 2, \ldots$	$\frac{pe^t}{1-(1-p)e^t}$	$\frac{1}{p}$	$\frac{1-p}{p^2}$

Table 2.2

Continuous probability distribution	Probability density function, $f(x)$	Moment generating function, $\phi(t)$	Mean	Variance
Uniform over (a, b)	$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise} \end{cases}$	$\frac{e^{tb} - e^{ta}}{t(b-a)}$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential with parameter $\lambda > 0$	$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x > 0 \\ 0, & x < 0 \end{cases}$	$\frac{\lambda}{\lambda - t}$	$\frac{1}{\lambda}$	$\frac{1}{\lambda^2}$
Gamma with parameters (n, λ), $\lambda > 0$	$f(x) = \begin{cases} \frac{\lambda e^{-\lambda x} (\lambda x)^{n-1}}{(n-1)!}, & x \ge 0 \\ 0, & x < 0 \end{cases}$	$\left(\frac{\lambda}{\lambda - t}\right)^n$	$\frac{n}{\lambda}$	$\frac{n}{\lambda^2}$
Normal with parameters (μ, σ^2)	$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \times \exp\{-(x-\mu)^2/2\sigma^2\}$, $-\infty < x < \infty$	$\exp\left\{\mu t + \frac{\sigma^2 t^2}{2}\right\}$	μ	σ^2