Representations of finite groups (WISB324)

Exam June 28 2017.

1a $\rho(a^n)=1$ and $\rho(b^2)=1$ are obvious, we only have to check the relation $\rho(bab)=\rho(a^{n-1})=\rho(a^{-1})=\rho(a)^{-1}$

$$\rho(bab)(x_i)=\rho(b)\rho(a)\rho(b)(x_i)$$
$$=\rho(b)\rho(a)(x_{n+1-i})$$
$$=\rho(b)\,x_{n+2-i\,(\mathrm{mod}\,n)}$$
$$=x_{i-1\,(\mathrm{mod}\,n)}\quad=\rho(a)^{-1}=\rho(a^{n-1}).$$

b The degree of a monomial does not change under the action of an element of D_{2n}. Hence the degree of a homogeneous polynomial does not change. Conclusion V_m is an invariant subspace of $\mathbb{C}[x_1,\dots x_n]$ and hence a $\mathbb{C}D_{2n}$-submodule

c $\rho(g)(x_1^m+x_2^m+\dots+x_n^m)=x_1^m+\dots+x_n^m$, hence

$\langle x_1^m+\dots+x_n^m\rangle$ is an invariant subspace and hence a $\mathbb{C}D_{2n}$-submodule. Thus V_m is not irreducible.

d Let $\omega=e^{\frac{2\pi i}{n}}$ and define $v_j=\sum_{k=1}^{n}\omega^{kj}x_k$ and

$W_j=\langle v_j\rangle$, then $\rho(a)W_j\subset W_j$, since $\rho(a)v_j=\omega^{-j}v_j$

and $\rho(b)W_j=W_{n-j}$ since $\rho(b)v_j=\omega^{j}v_{n-j}$.

Hence the irreducible submodules are W_n (recall n odd) and $W_k\oplus W_{n-k}$ $\quad k=1,2,\dots,\frac{n-1}{2}$.

2 Note that $\psi(g) = \omega_g$ for some k-th root of unity

(a) Now $1 = \langle \chi, \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi(g)}$

then $\langle \psi\chi, \psi\chi \rangle = \frac{1}{|G|} \sum_{g \in G} \psi(g)\chi(g) \overline{\psi(g)\chi(g)}$

$$= \frac{1}{|G|} \sum_{g \in G} \underbrace{\omega_g \overline{\omega_g}}_{=1} \chi(g) \overline{\chi(g)}$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\chi(g)} = \underbrace{\langle \chi, \chi \rangle}_{=1}$$

Since $\langle \psi\chi, \psi\chi \rangle = 1$, $\psi\chi$ is irreducible.

Since both χ and $\psi\chi$ are irreducible and degree $\psi\chi$ = degree χ. Both χ and $\psi\chi$ are irreducible characters of degree n. Since there is only one such character, $\psi\chi = \chi$.

(b) $\chi(g) = \psi(g)\chi(g)$, thus

$(1 - \psi(g))\chi(g) = 0$. Now if $\psi(g) \neq 0$ then

$\chi(g) = 0$.

3 Elements of G are 1, a^k, b^ℓ or $b^\ell a^k$ with $k < 7$, $\ell < 6$.

Now $b^{-\ell} a^p b^\ell = a^{3p\ell} \in M$

$(b^\ell a^k)^{-1} a^p b^\ell a^k = a^{-k} b^{-\ell} a^p b^\ell a^k =$

$a^{-k} a^{3p\ell} a^k = a^{3p\ell} \in M.$

Hence M is a normal subgroup. $G/M = \langle \bar{b} \rangle$ which is abelian.

(b) $b^{-1}ab = a^3$, $b^{-1}a^3b = a^2$, $b^{-1}a^2b = a^6$, $b^{-1}a^6b = a^4$

$b^{-1}a^4b = a^5$ $b^{-1}a^5b = a$.

Hence $\{1\}$ and $M\setminus\{1\}$ are two conjugacy classes.

Now ~~$b^{-1}a^k b^l$~~ $b^{-1}(b^l a^k)b = (b^{-1}a^k b)b^l$ and

$ab^l a^{-1} = b^l b^{-l} a b^l a^{-1} = b^l a^{3l-1}$

If $l \neq 5$ $3l-1 \not\equiv 0 \pmod 7$ and b^l is conjugate to all elements in $b^l M$.

If $l = 5$ then $a^2 b^5 a^{-2} = b^5 a$ hence b^5 is conjugate for all other elements in $b^5 M$.

Conjugacy classes	$\{1\}$,	$M\setminus\{1\}$,	bM,	b^2M,	b^3M,	b^4M,	b^5M.
representatives	1	a	b	b^2	b^3	b^4	b^5

(c) G/M is abelian, hence all irreducible characters of G/M are linear. $|G/M| = 6$, Hence there are 6 linear characters that we can lift to G. So G has 6 linear characters.
G has also 7 conjugacy classes and thus also 7 irreducible characters. Using the degree formula.

$$|G| = \sum_{j=1}^{7} d_j^2 = 1+1+1+1+1+1+d_7^2$$

We deduce that $d_7 = 6$.

(d) The 6 linear characters of G/M are defined on the generator $\bar{b}$ as

$$\chi_j(\bar{b}) = e^{\frac{2\pi i j}{6}}$$ we lift this to G.

which gives $\omega = e^{\frac{2\pi i}{6}}$

	1	a	b	b^2	b^3	b^4	b^5
χ_6	1	1	1	1	1	1	1
χ_1	1	1	ω	ω^2	ω^3	ω^4	ω^5
χ_2	1	1	ω^2	ω^4	1	ω^2	ω^4
χ_3	1	1	ω^3	1	ω^3	1	ω^3
χ_4	1	1	ω^4	ω^2	1	ω^4	ω^2
χ_5	1	1	ω^5	ω^4	ω^3	ω^2	ω
χ_7	6	-1	0	0	0	0	0

The zero's in The last row come from the fact that $\chi_1 \chi_7 = \chi_1$ (use Ex 2b). The -1 comes from the orthogonality of the first two columns.

(e) All normal subgroups come from intersections of kernels of linear characters, except for $\{e\}$. $\ker \chi_6 = G$, $\ker \chi_1 = \ker \chi_5 = M$.

$\mathrm{Ker}\, \chi_3 = M \cup b^2 M \cup b^4 M.$ and $\mathrm{Ker}(\chi_2) = \mathrm{Ker}\, \chi_4$
$= M \cup b^3 M.$

Taking intersections do not give more examples
Hence all normal subgroups are.

$M, \{e\}, G, M \cup b^3 M, M \cup b^2 M \cup b^4 M.$

(f) M is abelian and hence has only linear characters:

$\chi_\ell(a) = \varepsilon_\ell = e^{\frac{2\pi i}{7}\ell}$ Now let $\ell \neq 7$!

$\chi_\ell \uparrow G(g) = \sum_{j=0}^{5} \dot{\chi}(b^{-j} g\, b^j)$, now $b^{-j} g\, b^j \in M$
only if $g \in M$, hence $\chi_\ell \uparrow G(g) = 0$ if $g \notin M$
$\chi_\ell \uparrow g(1) = 6.$ since $\chi_\ell(b^{+j}\, 1\, b^j) = \chi(1)$
and $\chi_\ell \uparrow G(a) = \sum_{j=0}^{5} \chi(b^{-j} a\, b^j)$

$= \chi(a) + \chi(a^3) + \chi(a^2) + \chi(a^6) + \chi(a^4) + \chi(a^5)$

$= \varepsilon_\ell + \varepsilon_\ell^2 + \cdots + \varepsilon_\ell^6$

$= 1 + \varepsilon_\ell \quad + \varepsilon_\ell^7 - 1$

$= \dfrac{1-\varepsilon_\ell^7}{1-\varepsilon_\ell} - 1 \quad = -1$

Hence $\chi_\ell \uparrow G = \chi_7.$

4(a) $\Rightarrow$ If all characters are real then also all irred. characters are real.

$\Leftarrow$ A character X can be expressed in the irreducible characters $X_1, \dots, X_\ell$ as follows

$$X = \sum_{k=1}^{\ell} d_k X_\ell \quad \text{where } d_k \in \mathbb{Z} \text{ and } d_k \geq 0.$$

Hence, since all X_ℓ are real and all $d_\ell \in \mathbb{Z}$ also X is real $\square$.

(b) All automorphisms of C_p are defined by sending its generator x to x^ℓ with $\ell = 1, 2, \dots p-1$. Hence there are $p-1$ different automorphisms.

(c) $\rho_a \cdot \rho_b(x) = \rho_a(b x b^{-1}) = \cancel{\rho}\ a b x b^{-1} a^{-1} =$

$a b x (ab)^{-1} = \rho_{ab}(x)$.

$|G/C^p| = m \quad a \notin C^p \quad [a^m] = [e]$, thus $a^m \in C_p$.

$a^{mp} = 1$ thus a^m has order p or 1.

$a^m x a^{-m} = x a^m a^{-m} = x \quad$ since $a^m, x, a^{-m} \in C_p$.

Thus $\rho_{a^m} = \cancel{\rho a}(\rho_a)^m = 1$.

~~(d) From (b) and (c) it follows that ρ_a has order $p-1$ and order m. Since $\gcd(p-1, m) = 1$ ρ_a must have order~~

(d) From exercise b and c it follows that the order of ρ_a must divide both $p-1$ and m, but since $\gcd(p-1, m) = 1$ The order must be 1 and hence

$$\rho_a = 1.$$

(e) Note • $aga^{-1} \in C_p$ only if $g \in C_p$

• $aga^{-1} = g$ if $g \in C_p$.

Thus. $\varphi\uparrow G(g) = 0$ if $g \notin C_p$.

$$\varphi\uparrow G(g) = \frac{1}{P} \sum_{a \in G} \dot{\varphi}(a g a^{-1})$$

$$= \frac{1}{P} \sum_{a \in G} \dot{\varphi}(g)$$

$$= \begin{cases} 0 & \text{if } g \notin M \\ \frac{1}{|P|} |G| \varphi(g) = m\varphi(g) & \text{if } g \in M. \end{cases}$$

Since $p > 2$ there exists a character such that $\varphi(X) = e^{\frac{2\pi i}{p}} \notin \mathbb{R}$ for X the generator of C_p. Hence $\varphi\uparrow G(X) = m e^{\frac{2\pi i}{p}} \notin \mathbb{R}$.

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$$\langle \psi\uparrow G, \chi\rangle_G = \frac{1}{|G|}\sum_{g\in G} \psi\uparrow G(g)\,\overline{\chi(g)}$$

$$= \frac{1}{|G|}\sum_{g\in G}\frac{1}{|H|}\sum_{y\in G}\dot\psi(y^{-1}gy)\,\overline{\chi(g)}$$

$$\underset{\tilde g = y^{-1}gy}{=} \frac{1}{|G|}\cancel{\sum}\frac{1}{|H|}\sum_{\tilde g\in G}\sum_{y\in G}\dot\psi(\tilde g)\,\overline{\chi(y\tilde g y^{-1})}$$

$$\underset{*}{=} \frac{1}{|G|}\frac{1}{|H|}\sum_{\tilde g\in G}\sum_{y\in G}\dot\psi(\tilde g)\,\overline{\chi(\tilde g)}$$

$$= \frac{1}{|H|}\sum_{\tilde g\in G}\dot\psi(\tilde g)\,\overline{\chi(\tilde g)}$$

$$\underset{**}{=} \frac{1}{|H|}\sum_{h\in H}\psi(h)\,\chi(h)$$

$$= \langle \psi, \chi\downarrow H\rangle$$

$*$: $\chi(x) = \chi(g x g^{-1})$ constant on conjugacy classes

$**$: $\dot\psi(g) = \begin{cases} 0 & g \in H \\ \psi(g) & g\in H. \end{cases}$

$\square$