Inleiding Topologie exam, January 28, 2015

Note: In all the questions below, please explain/prove your answers (e.g., in Exercise 1, part a., b., etc, please do not just say "yes" or "no". Also, in part e. or g. of the same exercise, do not just write down the final result, but also explain how you found it- e.g. by explaining your reasoning using pictures). There are three questions marked with a "*"; they are probably more difficult than the rest. The final mark for the exam is the number of points that you collect, except for the case in which you collect more than 10 points, when the final mark will be 10 (in total, the exercises below are worth 11.5 points).

Exercise 1. Prove that, for any Hausdorff space $(X, \mathcal{T})$, any finite subset $F \subset X$ is closed in X.
(1 point)

Exercise 2. Consider the family $\mathcal{B}$ of subsets of $\mathbb{R}^2$ consisting of all the subsets of type $(a,b) \times (a,b)$ with $a < b$ real numbers:

$$\mathcal{B} = \{(a,b) \times (a,b) : a, b \in \mathbb{R}, a < b\}.$$

Let $\mathcal{T}$ be the smallest topology on $\mathbb{R}^2$ containing $\mathcal{B}$. We also consider

$$A = [0,1] \times [0,2] \subset \mathbb{R}^2.$$

a. Is $(\mathbb{R}^2, \mathcal{T})$ second countable? *(0.5 points)*

b. Is $(\mathbb{R}^2, \mathcal{T})$ Hausdorff? *(0.5 points)*

c. Is the identity map $\mathrm{Id} : \mathbb{R}^2 \to \mathbb{R}^2$ continuous as a map from $(\mathbb{R}^2, \mathcal{T})$ to $(\mathbb{R}^2, \mathcal{T}_{\mathrm{Eucl}})$? But as a map from $(\mathbb{R}^2, \mathcal{T}_{\mathrm{Eucl}})$ to $(\mathbb{R}^2, \mathcal{T})$? *(0.5 points)*

d. Is A, with the topology induced from $\mathcal{T}$, connected? Is it compact? *(0.5 points)*

e. Let $x = (0,2) \in \mathbb{R}^2$ (the point of coordinates 0 and 2). Compute the closure of $\{x\}$ in $(\mathbb{R}^2, \mathcal{T})$. *(0.5 points)*

f. Show that the sequence $(x_n)_{n \geq 1}$ given by

$$x_n = (sin^2(n), cos^4(n + 2015)) \in \mathbb{R}^2$$

is convergent in $(\mathbb{R}^2, \mathcal{T})$ and has more than one limit. *(0.5 points)*

g. Compute the interior and the closure of A in $(\mathbb{R}^2, \mathcal{T})$. *(1 point)*

h*. Show that any continuous map $f : (\mathbb{R}^2, \mathcal{T}) \to \mathbb{R}$ must be constant. *(1 point)*

(total: $6 \times 0.5 + 2 \times 1 =$ **5 points**)

Exercise 3. Consider the group of integers modulo 2, $\mathbb{Z}_2 = \{\hat{0}, \hat{1}\}$. Define the following action of $\mathbb{Z}_2$ on the closed unit disk $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$:

$$\hat{0} \cdot z = z, \ \hat{1} \cdot z = -z \quad \text{(for all } z \in D^2\text{)}.$$

Prove that the resulting quotient $D^2/\mathbb{Z}_2$ is homeomorphic to D^2 (provide a complete argument; a "proof" based only on pictures is not enough to get the entire 1 point for this exercise).
(**1 point**)

Exercise 4. Let X be compact Hausdorff space and let $C(X)$ be the space of real-valued continuous functions on X, endowed with the topology induced by the *sup*-metric.

Prove that if $\mathcal{A} \subset C(X)$ is a dense subset of $C(X)$, then $\mathcal{A}$ must be point-separating.
(1 point)

Exercise 5. For any algebra A over $\mathbb{R}$ and any ideal I of A, we define

$$X(A, I) := \{\chi \in X_A : \chi(x) = 0 \ \ \forall\, x \in I\} \subset X_A$$

and we endow it with the topology induced from the topology on the spectrum X_A of A. Show that:

a. For any algebra A and any ideal I of A, $X(A, I)$ is closed in X_A. *(0.5 points)*

b. Applied to $A = \mathbb{R}[x, y]$ (polynomials in two variables) and I the ideal consisting of polynomials that are are divisible by $x^2 + y^2 - 1$, $X(A, I)$ is homeomorphic to S^1. *(0.5 points)*

c.* Assume now that X is a compact Hausdorff space, $Y \subset X$ and set

$$A = C(X), \quad I := \{f \in C(X) : f(y) = 0 \ \ \forall\, y \in Y\}.$$

Show that $X(A, I)$ is homeomorphic to the closure $\overline{Y}$ of Y in X (where the last space is endowed with the topology induced from X). *(1 point)*

(total: $2 \times 0.5 + 1 =$ **2 points**)

Exercise 6. For each natural number n we consider a space X_n that is obtained by removing n distinct points from $\mathbb{R}^2$. We consider the 1-point compactification X_n^+ and we denote by $\infty_n \in X_n^+$ the point at infinity (so that $X_n^+ = X_n \cup \{\infty_n\}$). Show that

a. X_n^+ can be embedded in $\mathbb{R}^3$ (here you do not have to write down explicit formulas for the embedding, but please explain your reasoning using pictures and mention what result(s) you use in order to reach the final conclusion). *(0.5 points)*

b*. If X_n and X_m are homeomorphic, then $n = m$. *(1 point)*

(total: $0.5 + 1 =$ **1.5 points**)

END EXAM

Exercise 7. For each natural number n we denote by X_n the space obtained after removing n distinct points from $\mathbb{R}^2$. We consider the 1-point compactification X_n^+ Show that, if $n \neq m$, then X_n is not homeomorphic to X_m.

Exercise 8. Assume that

$$f : \mathbb{R} \to [0, \infty)$$

is a continuous map with the property that the equation

$$f(x) = 0$$

has precisely n distinct solutions. Show that there exists $\epsilon > 0$ such that, for any $c \in (0, \epsilon)$, the equation

$$f(x) = c$$

has at least $2n$ solutions.

Exercise 9. Let A be an algebra over $\mathbb{R}$, assume that $\chi : A \to \mathbb{R}$ is a character and let

$$I := \mathrm{Ker}(\chi)(= \{a \in A : \chi(a) = 0\}.$$

Show that:

a. I is an ideal of A.

b. $a - \chi(a) \cdot \mathbb{1}_A \in I$ for all $a \in A$ (where $\mathbb{1}_A$ is the unit of A).

c. I is a maximal ideal.

Exercise 10. Let X be a compact Hausdorff space,

$$F = (F_1, \ldots, F_n) : X \to \mathbb{R}^n$$

a continuous function. Denote by $\mathcal{A}_F$ the set of functions $f : X \to \mathbb{R}$ with the property that there exists a polynomial P in n variables such that

$$f(x) = P(F_1(x), \ldots, F_n(x)) \;\; \forall \, x \in X.$$

Show that F is an embedding if and only if $\mathcal{A}_F$ is dense in $C(X)$.

Exercise 11. Consider the group $\mathbb{Z}_2$ (integers modulo 2, with addition modulo 2 as operation) and look at actions of $\mathbb{Z}_2$ on the unit circle S^1 which are different from the trivial action (where the trivial action is the one given by $\hat{k} \cdot z = z$ for all $\hat{k} \in \mathbb{Z}_2$ and $z \in S^1$).

1. describe an action of $\mathbb{Z}_2$ on S^1 so that $S^1/\mathbb{Z}_2$ is homeomorphic to $[0, 1]$.
2. describe a non-trivial action of $\mathbb{Z}_2$ on S^1 so that $S^1/\mathbb{Z}_2$ is homeomorphic to S^1.
3. describe an action of $\mathbb{Z}_2$ on S^1 so that $S^1/\mathbb{Z}_2$ is homeomorphic to $[0, 1)$.

(total: $3 \times 0.5 =$ **1.5 points**)