Utrecht University
Introductory mathematics for finance
WISB373
Winter 2015

Exam May 27, 2015

JUSTIFY YOUR ANSWERS
Allowed material: calculator, material handed out in class and *handwritten* notes (*your handwriting*). NO BOOK IS ALLOWED

NOTE: The test consists of four questions for a total of 10 points

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Exercise 1. A *saving plan* is a sequence of n yearly payments for an amount C. At the end of the last payment the saver collects some good (car, house, lump sum of money) for the total value P_n of all the payments at the final time. The yearly (simple or effective) interest rate is r.

(a) (0.7 pts.) Prove that this final value is

$$P_n \;=\; C\,\frac{(1+r)^n-1}{r}\;.$$

(b) (0.7 pts.) You subscribe a saving plan for 10 years at a yearly interest of 5%. How many more years you should continue paying if you want at the end to collect twice P_{10}.

Exercise 2. ***(Discrete stochastic integrals and sub-martingales)*** Let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration on a probability space and let $(D_n)_{n\geq 0}$ and $(W_n)_{n\geq 0}$ be adapted processes. Let $(Y_n)_{n\geq 0}$ be the process defined by

$$Y_n \;=\; W_0 + \sum_{\ell=1}^{n} D_{\ell-1}\,\big(W_\ell - W_{\ell-1}\big) \tag{1}$$

Prove the following

(a) (0.8 pts.) If $(W_n)_{n\geq 0}$ is a martingale, then so is $(Y_n)_{n\geq 0}$.

(b) (0.8 pts.) If $D_n \geq 0$, then

-i- (0.8 pts.) $(Y_n)_{n\geq 0}$ is a super-martingale if so is $(W_n)_{n\geq 0}$.

-ii- (0.8 pts.) If $W_0 \geq 0$ and $(W_n)_{n\geq 0}$ is a sub-martingale, then so is $(Y_n^2)_{n\geq 0}$. [*Hint:* Use Jensen's inequality.]

Exercise 3. [European option with variable interest] A stock whose present value is $S_0 = 4$ evolves following a binomial model with $u = 1.5$ and $d = 1/2$; both possibilities having equal probability. The interest rate for the initial period is 5%, in each subsequent i-th period the interest jumps to 10% if $\omega_i = H$ and reverts to 5% if $\omega_i = T$. A European call option is established for 3 periods with strike value $K = S_0$ and payoff

$$V_3 \;=\; \big|S_3 - S_0\big|_+\;.$$

Determine

(a) (0.7 pt.) Determine the risk-neutral probability for three periods.

(b) (0.8 pts.) The fair price of the option.

(c) (0.8 pts.) The hedging strategy for the seller.

(d) (0.8 pts.) The owner of the option decides to sell it at the end of the first period. Find the fair value for both values of ω_1.

Exercise 4. [American option] Consider the same stock evolution as in the previous exercise, but with a constant interest of 5%. An American put option is established for 3 periods with strike value $K = S_0$, intrinsic payoff

$$G_n = S_0 - S_n \quad , \quad n = 0, 1, 2 \, ,$$

and final payoff

$$V_3 = \big|S_0 - S_3\big|_+ \, . \tag{2}$$

(a) (0.8 pts.) Determine the fair price V_0 of the option.

(b) (0.8 pts.) The optimal exercise time τ^* for the buyer.

(c) (0.8 pts.) Show that V_0 and τ^* satisfy the identity

$$V_0 = \widetilde{E}\Big[\mathbb{I}_{\{\tau^* \leq 3\}} \, \frac{G_{\tau^*}}{R_0 \cdots R_{\tau^*-1}}\Big] \, ,$$

(d) (0.7 pts.) Prove that the fair value of the preceding American option is larger or equal than the value of an European option with the same payoff (2).