Measure and Integration: Hertentamen 2014-15

(1) Consider the measure space $([0,1), \mathcal{B}([0,1)), \lambda)$, where $\mathcal{B}([0,1))$ is the Borel σ-algebra restricted to $[0,1)$ and λ is the restriction of Lebesgue measure on $[0,1)$. Define the transformation $T : [0,1) \to [0,1)$ given by

$$T(x) = \begin{cases} 3x & 0 \le x < 1/3, \\ 3x-1, & 1/3 \le x < 2/3 \\ 3x-2, & 2/3 \le x < 1. \end{cases}$$

(a) Show that T is $\mathcal{B}([0,1))/\mathcal{B}([0,1))$ measurable. (0.5 pts)
(b) Determine the image measure $T(\lambda) = \lambda \circ T^{-1}$. (0.5 pts)
(c) Show that for all $f \in \mathcal{L}^1(\lambda)$ one has, $\int f\,d\lambda = \int f \circ T\,d\lambda$. (0.5 pts)
(d) Let $\mathcal{C} = \{A \in \mathcal{B}([0,1)) : \lambda(T^{-1}A \Delta A) = 0\}$. Show that $\mathcal{C}$ is a σ-algebra. (0.5 pts)

(2) Consider the measure space $((0,\infty), \mathcal{B}((0,\infty)), \lambda)$, where $\mathcal{B}((0,\infty))$ is the restriction of the Borel σ-algebra, and λ Lebesgue measure restricted to $(0,\infty)$. Determine the value of

$$\lim_{n\to\infty} \int_{(0,n)} \frac{\cos(x^5)}{1+nx^2}\,d\lambda(x).$$

(2 pts)

(3) Let $(X, \mathcal{A}, \mu)$ be a finite measure space, and $1 < p, q < \infty$ two conjugate numbers (i.e. $1/p + 1/q = 1$). Let $g \in \mathcal{M}(\mathcal{A})$ be a measurable function satisfying

$$\int |fg|\,d\mu \le C||f||_p$$

for all $f \in \mathcal{L}^p(\mu)$ and for some constant C.
(a) For $n \ge 1$, let $E_n = \{x \in X : |g(x)| \le n\}$ and $g_n = \mathbf{1}_{E_n}|g|^{q/p}$. Show that $g_n \in \mathcal{L}^p(\mu)$ for all $n \ge 1$. (0.5 pts)
(b) Show that $g \in \mathcal{L}^q(\mu)$. (1.5 pts)

(4) Let $(X, \mathcal{A}, \mu)$ be a σ-finite measure space, and (f_j) a uniformly integrable sequence of measurable functions. Define $F_k = \sup_{1\le j\le k}|f_j|$ for $k \ge 1$.
(a) Show that for any $w \in \mathcal{M}^+(\mathcal{A})$,

$$\int_{\{F_k > w\}} F_k\,d\mu \le \sum_{j=1}^{k} \int_{\{|f_j|>w\}} |f_j|\,d\mu.$$

(0.5 pts)
(b) Show that for every $\epsilon > 0$, there exists a $w_\epsilon \in \mathcal{L}^1_+(\mu)$ such that for all $k \ge 1$

$$\int_X F_k\,d\mu \le \int_X w_\epsilon\,d\mu + k\epsilon.$$

(1 pt)
(c) Show that

$$\lim_{k\to\infty} \frac{1}{k} \int_X F_k\,d\mu = 0.$$

(0.5 pts)

(5) Consider the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda)$, where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra, and λ Lebesgue measure. Let $k, g \in \mathcal{L}^1(\lambda)$ and define $F : \mathbb{R}^2 \to \mathbb{R}$, and $h : \mathbb{R} \to \overline{\mathbb{R}}$ by

$$F(x,y) = k(x-y)g(y).$$

(a) Show that F is measurable. (1 pt)

(b) Show that $F \in \mathcal{L}^1(\lambda \times \lambda)$, and

$$\int_{\mathbb{R}\times\mathbb{R}} F(x,y)d(\lambda \times \lambda)(x,y) = \left(\int_{\mathbb{R}} k(x)d\lambda(x)\right)\left(\int_{\mathbb{R}} g(y)d\lambda(y)\right).$$

(1 pts)