Utrecht University
Mathematical Institute

**Final Exam for Introduction to Financial Mathematics, WISB373**

Monday January 30 2023, 13:30-16:30 o'clock **(3 hours examination)**

**1.** Answer the statements below by either TRUE or FALSE. In case of "TRUE", explain your answer. In case of "FALSE", explain why the statement is false and give the correct statement.

a. An option gives the holder the right to buy or sell an option from a writer at time $t = T$.

b. The put-call parity is a relation between call and put option prices that should hold at any time $t \in [0, T]$, with $T$ the expiry date.

c. The Black-Scholes price is the fair price of a European option when the stock price is modeled by a Geometric Brownian Motion.

d. A European option is only traded on European exchanges, while an American option is only traded in North-, South- and Central America.

*Answer:* a. False, because an option gives the right to buy or sell STOCK at time $T$

b. True, the put-call relation should hold for any time $t$ as otherwise there is an arbitrage opportunity

c. True; under GBM we can deduce the fact that call and puts at any time $0 \leq t \leq T$ for any $S(t)$ can be found by means of the solution of the B-S PDE, giving rise to the B-S solution.

d. False, these are traded everywhere. Eurpoean or American is not related to the continent where they are being traded.

**2.** Let $W(t)$ be a standard Brownian motion. Evaluate

$$\int_0^T W^2(t)\mathrm{d}t + \int_0^T 2\,t\,W(t)\mathrm{d}W(t).$$

Which of the answers below is correct,

I) $0$,

II) $W^3(T)/3 + T^2W^2(T) - T^2/2$,

III) $2W^3(T)/3 + T^2W(T) - T^2$,

IV) $TW^2(T) - T^2/2$,

V) $2TW^2(T) - T$.

Justify your answer.

*Answer:* The correct answer is IV!

Let $f(W,t) = tW^2$. Then the Itô-Doeblin formula yields

$$\begin{aligned} \mathrm{d}(f(W,t)) &= f_t\mathrm{d}t + f_W\mathrm{d}W + (1/2)f_{WW}\mathrm{d}t, \\ &= W^2dt + 2tW\mathrm{d}W + (1/2)2t\mathrm{d}t, \\ &= (W^2(t) + t)\mathrm{d}t + 2tW(t)\mathrm{d}W(t). \end{aligned}$$

Rearranging we have

$$W^2(t)\mathrm{d}t + 2tW(t)\mathrm{d}W(t) = d(tW^2(t)) - t\mathrm{d}t.$$

Integrating from 0 to $T$ yields

$$\begin{aligned}
\int_0^T W^2(t)\mathrm{d}t + \int_0^T 2tW(t)\mathrm{d}W(t) &= \int_0^T d(tW^2(t)) - \int_0^T t\mathrm{d}t. \\
&= (tW(t)|_0^T - \frac{1}{2}T^2 \\
&= TW^2(T) - \frac{1}{2}T^2.
\end{aligned}$$

**3.** Let $\theta \in [0, 2\pi)$ and $W = (W_1, W_2)$ be a two-dimensional Brownian motion. Use Lévy's characterisation of Brownian motion to show that if

$$Y_1(t) = \cos(\theta)W_1(t) - \sin(\theta)W_2(t), \quad Y_2(t) = \sin(\theta)W_1(t) + \cos(\theta)W_2(t)$$

then $Y = (Y_1, Y_2)$ is a two-dimensional Brownian motion.

*Answer:* Clearly $Y$ is adapted and $Y(0) = 0$ since $W_1(0) = W_2(0) = 0$ Moreover, $Y_1$ and $Y_2$ are linear combinations of continuous martingales and therefore themselves continuous martingales. So, $Y$ is a two-dimensional martingale. It remains to check that $[Y_1, Y_1(t)] = [Y_2, Y_2](t) = t$ and $[Y_1, Y_2](t) = 0$ for all $t \geq 0$. By linearity, and then symmetry, of the quadratic variation:

$$\begin{aligned}
[Y_1, Y_1](t) &= [\cos(\theta)W_1, \cos(\theta)W_1](t) - [\cos(\theta)W_1, \sin(\theta)W_2](t) \\
&\quad -[\sin(\theta)W_2, \cos(\theta)W_1](t) + [\sin(\theta)W_2, \sin(\theta)W_2](t) \\
&= \cos^2(\theta)[W_1, W_1](t) - 2\cos(\theta)\sin(\theta)[W_1, W_2](t) + \sin^2(\theta)[W_2, W_2](t) \\
&= (\cos^2(\theta) + \sin^2(\theta))t = t.
\end{aligned}$$

The computation showing $[Y_2, Y_2](t) = t$ is similar. Finally,

$$\begin{aligned}
[Y_1, Y_2](t) &= \cos(\theta)\sin(\theta)[W_1, W_1](t) + (\cos^2(\theta) + \sin^2(\theta))[W1, W2](t) \\
&\quad - \sin(\theta)\cos(\theta)[W_2, W_2](t) = 0.
\end{aligned}$$

**4.** Let $\{W(t) : t \geq 0\}$ be a Brownian motion, and define the process $\{X(t) : t \geq 0\}$ as

$$X(t) = W^3(t) + \ c\, t\, W(t).$$

For what value of $c$ is $X(t)$ a martingale? Find a process $\{\Gamma(t), t \geq 0\}$ adapted to $\{\mathcal{F}(t), t \geq 0\}$ such that

$$X(t) = X(0) + \int_0^t \Gamma(s)\mathrm{d}W(s).$$

*Answer:* $c = -3$, which can be found as follows:

Itô's lemma states that for any process $Y(t)$,

$$\mathrm{d}(f(Y, t)) = f_t\mathrm{d}t + \frac{1}{2}f_{YY}(\mathrm{d}Y)^2 + f_Y\mathrm{d}Y.$$

With $W(t) = Y$, $X(t) = f(W, t) = W^3(t) + ct$, we have

$$f_t = cW(t), \ f_W = 3W^2(t) + ct, \text{ and } f_{WW} = 6W(t).$$

Therefore:

$$\begin{aligned}
\mathrm{d}X(t) &= cW(t)\mathrm{d}t + \frac{1}{2}(6W(t))\mathrm{d}t + (3W^2(t) + ct)\mathrm{d}W(t) \\
&= (c + 3)W(t)\mathrm{d}t + (3W^2(t) + ct)\mathrm{d}W(t).
\end{aligned}$$

Setting the coefficients of $dt = 0$. gives us, $c = -3$. In that case the process $\Gamma(t) = (3\mathrm{W}^2(t) - 3t)$.

**5.** Theorem: Let $\{X_1(t), t \geq 0\}$ and $\{X_2(t), t \geq 0\}$ be the diffusion processes

$$\mathrm{d}X_i(t) = \alpha_i(t)\mathrm{d}t + \sigma_i(t)\mathrm{d}W(t).$$

Then $\{X_1(t)X_2(t), t \geq 0\}$ is the diffusion process given by

$$\mathrm{d}(X_1(t)X_2(t)) = X_2(t)\mathrm{d}X_1(t) + X_1(t)\mathrm{d}X_2(t) + \sigma_1(t)\sigma_2(t)dt.$$

Prove the theorem in the case that $\alpha_i$ and $\sigma_i$ are deterministic constants and $X_i(0) = 0$, for $i = 1, 2$.

*Answer:* Assume

$$X_i(t) = \alpha_i\mathrm{d}t + \sigma_i\mathrm{d}W(t), i = 1, 2,$$

for constants $\alpha_1, \alpha_2, \sigma_1, \sigma_2$. Then, the right hand side of the integral form of the requested equality is

$$\begin{aligned}
&\int_0^t [(\alpha_2 s + \sigma_2 W(s))\sigma_1 + (\alpha_1 s + \sigma_1 W(s))\sigma_2]\mathrm{d}W(s) \\
&\int_0^t [(\alpha_2 s + \sigma_2 W(s))\alpha_1 + (\alpha_1 s + \sigma_1 W(s))\sigma_2]\alpha_2 + \sigma_1\sigma_2]\mathrm{d}s \\
&= \sigma_1 \int_0^t [(\alpha_2 s + \sigma_2 W(s))\mathrm{d}W(s) + \sigma_2 \int_0^t (\alpha_1 s + \sigma_1 W(s))\mathrm{d}W(s) \\
&+\alpha_1\alpha_2 \frac{t^2}{2} + \alpha_1\sigma_2 \int_0^t W(s)\mathrm{d}s + \alpha_1\alpha_2 \frac{t^2}{2} + \sigma_1\alpha_2 \int_0^t W(s)\mathrm{d}s + \sigma_1\sigma_2 t \\
&= \sigma_1\alpha_2 \int_0^t s\mathrm{d}W(s) + 2\sigma_1\sigma_2 \int_0^t W(s)\mathrm{d}W(s) + \sigma_2\alpha_1 \int_0^t s\mathrm{d}W(s) \\
&+\alpha_1\alpha_2 t^2 + \sigma_1\sigma_2 t + (\alpha_1\sigma_2 + \sigma_1\alpha_2) \int_0^t W(t)\mathrm{d}s \\
&= 2\sigma_1\sigma_2(\frac{W^2(t)}{2} - \frac{t}{2}) + \sigma_1\sigma_2 t + \alpha_1\alpha_2 t^2 + (\sigma_1\alpha_2 + \alpha_1\sigma_2)\left(\int_0^t s\mathrm{d}W(s) + \int_0^t W(s)\mathrm{d}s\right) \\
&= \sigma_1\sigma_2 W^2(t) + \alpha_1\alpha_2 t^2 + (\sigma_1\alpha_2 + \alpha_1\sigma_2)tW(t) \\
&= (\alpha_1 t + \sigma_1 W(t))(\alpha_2 t + \sigma_2 W(t)) = X_1(t)X_2(t).
\end{aligned}$$

**6.** Given the following linear stochastic differential equation for $\{X(t), t \geq 0\}$:

$$\mathrm{d}X(t) = tX(t)\mathrm{d}t + \mathrm{d}W(t),\ t \geq 0$$

with initial value $X(0) = 1$.

**a.** With $Y(t) = \exp(-t^2/2) \cdot X(t)$, derive the dynamics of $Y(t)$ and find $Y(0)$.

**b.** Find the solution $\{X(t), t \geq 0\}$ and determine its distribution.

*Answer:* Letting $Y(t) = \mathrm{e}^{-\frac{t^2}{2}}X(t)$, we find that $\mathrm{d}Y(t) = \mathrm{e}^{-\frac{t^2}{2}}\mathrm{d}W(t)$ and $Y(0) = 1$. Thus

$$X(t) = \mathrm{e}^{\frac{t^2}{2}} + \mathrm{e}^{\frac{t^2}{2}} \int_0^t \mathrm{e}^{-\frac{u^2}{2}}\mathrm{d}W(u).$$

Note that $X(t)$ is normally distributed with mean $\mathbb{E}[X(t)] = \mathrm{e}^{-\frac{t^2}{2}}$.

**7.** Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ its natural filtration, and $\mathcal{F} = \mathcal{F}(T)$. Consider a stock price process $\{S(t) : 0 \leq t \leq T\}$ with $S(t) = t^3 + 3W(t)$.

**a.** Construct a measure $\tilde{\mathbb{P}}$ equivalent to $\mathbb{P}$ (i.e., $\tilde{\mathbb{P}}(A) = 0$ if and only if $\mathbb{P}(A) = 0,\ A \in \mathcal{F}$), such that the price process $\{S(t) : 0 \le t \le T\}$ is a martingale under $\tilde{\mathbb{P}}$ and with respect to the filtration $\{\mathcal{F}(t) : 0 \le t \le T\}$. (10 pts.)

**b.** Consider a European call option on this stock with expiration date $T$ and strike price $K$. Find an expression for

$$C(0) = \tilde{\mathbb{E}}[\mathrm{e}^{-rT}(S(T) - K)^+],$$

the price of this option at time 0, with constant interest rate $r$.

Proof **3(a)**: Define $\theta(t) = t^2$, then

$$\frac{S(t)}{3} = \int_0^t \theta(u)\mathrm{d}u + W(t).$$

Consider the random variable $Z$ ($= Z(T)$, as given in Girsanov's Theorem), defined by

$$Z = \exp\left(-\int_0^T \theta(u)\mathrm{d}W(u) - \frac{1}{2}\int_0^T \theta^2(u)\mathrm{d}u\right) = \exp\left(-\int_0^T u^2\mathrm{d}W(u) - \frac{T^5}{10}\right).$$

Note that $\int_0^t \theta(u)\mathrm{d}W(u), \int_0^t \theta^2(u)\mathrm{d}u$ and $\theta$ are continuous functions on the compact interval $[0, T]$, hence they are all bounded and the same holds for $Z$. This implies that $\mathbb{E}\left[\int_0^T \theta^2(u)Z^2(u)\mathrm{d}u\right] < \infty$. Define the measure $\tilde{\mathbb{P}}$ on $\mathcal{F}$ by $\tilde{\mathbb{P}}(A) = \int_A Z$. By Girsanov's Theorem, the process $\{\tilde{W}(t) = \frac{S(t)}{3} : 0 \le t \le T\}$ is a Brownian motion under $\tilde{\mathbb{P}}$ and hence it is a martingale under $\tilde{\mathbb{P}}$. Since multiplying a martingale with a constant remains a martingale, we see that $\{S(t) : 0 \le t \le T\}$ is a martingale under $\tilde{\mathbb{P}}$.

Proof **3(b)**: By part (a), we see that under the measure $\tilde{\mathbb{P}}$, the random variable $S(T)/3$ is $N(0, T)$ distributed, hence $S(T)$ is $N(0, 9T)$, and

$$\begin{aligned}
c(0) &= \tilde{\mathbb{E}}\left[\mathrm{e}^{-rT}(S(T) - K)^+\right] \\
&= \mathrm{e}^{-rT}\int_{-\infty}^{\infty}(x - K)^+\frac{1}{\sqrt{18\pi T}}\mathrm{e}^{-\frac{x^2}{18T}}\mathrm{d}x \\
&= \mathrm{e}^{-rT}\int_K^{\infty}(x - K)\frac{1}{\sqrt{18\pi T}}\mathrm{e}^{-\frac{x^2}{18T}}\mathrm{d}x \\
&= \mathrm{e}^{-rT}\int_{K/3\sqrt{T}}^{\infty}(3\sqrt{T}y - K)\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}} \\
&= \mathrm{e}^{-rT}\int_{K/3\sqrt{T}}^{\infty}3\sqrt{T}y\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}} - Ke^{-rT}\int_{K/3\sqrt{T}}^{\infty}\frac{1}{\sqrt{2\pi}}\mathrm{e}^{-\frac{y^2}{2}} \\
&= \mathrm{e}^{-rT-K^2/18T}3\sqrt{\frac{T}{2\pi}} - \mathrm{e}^{-rT}K\left(1 - N(K/3\sqrt{T})\right)
\end{aligned}$$

where $N(y)$ is the standard normal distribution function.

**8.** The price of a call option under Geometric Brownian Motion stock dynamics, with exercise price $K$ and expiry date $T$, time $t$ and stock price $S(t)$ is given by:

$$c(S, t) = SN(d_+) - Ke^{-r(T-t)}N(d_-),$$

where

$$d_\pm = \frac{\log(S/K) + (r \pm \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}},$$

and

$$N(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy$$

is the $N(0,1)$ CDF. Use the put-call parity to show that the corresponding put option value is given by:

$$p(S,t) = Ke^{-r(T-t)} N(-d_-) - SN(-d_+).$$

**Please, make sure that your name is written down on each of the submitted solutions.**