Utrecht University
Mathematical Institute

Re-Examination for Introduction to Financial Mathematics, WISB373

Monday March 13th 2023, 17:00-20:00 o'clock **(3 hours examination)**

(Each item is worth 10 points)

1. Assume we have a European call $c(t)$ and a put option $p(t)$, with the same expiry date $T = 4$, i.e., exercise in 4 years, and strike price $K = 10$ Euro. The current share price is 11 Euro, assuming a zero constant interest rate $r = 0\%$. Determine if there exists an arbitrage opportunity if both options currently have the value $c(0) = 2.5$ Euro and $p(0) = 1.5$ Euro.

2. **a.** The random process $Y(t)$ is defined as $Y(t) = \alpha W(t) - \sqrt{\beta} W^*(t)$, where $W(t)$ and $W^*(t)$ are independent standard Brownian motions. Determine the relationship between α and β for which $Y(t)$ is a Brownian motion.

b. Determine whether $Z(t) = W(t) + 4t$ is a martingale.

c. Let $v_1, v_2, v_3 \in \mathbb{R}^3$ be orthonormal vectors, i.e. $v_i \cdot v_j = \delta_{ij}$. If $W(t) = (W_1(t), W_2(t), W_3(t))$ is a three-dimensional Brownian motion and $X_j(t) = v_j \cdot W(t)$ for $j \in \{1, 2, 3\}$, show, with the help of Lévy's characterization, that (X_1, X_2, X_3) is another three-dimensional Brownian motion.

3. Let $Q(t)$ denote the exchange rate at time t. It is the price in domestic currency of one unit of foreign currency and converts foreign currency into domestic currency. A model for the dynamics of the exchange rate is

$$\frac{\mathrm{d}Q(t)}{Q(t)} = \mu_Q \mathrm{d}t + \sigma_Q \mathrm{d}W(t).$$

This has the same structure as the common model for the stock price. The reverse exchange rate, denoted $R(t)$, is the price in foreign currency of one unit of domestic currency $R(t) = 1/Q(t)$. Derive $\mathrm{d}R(t)$

4. Let $\{W(t) : t \geq 0\}$ be a Brownian motion with filtration $\{\mathcal{F}(t) : t \geq 0\}$. Let $Y(t) = \int_0^t W^2(u)\mathrm{d}W(u) - \frac{1}{2}\int_0^t W^4(u)\mathrm{d}u$ and $X(t) = e^{Y(t)}$, for $t \geq 0$.

a. Prove that $X(t) = 1 + \int_0^t X(u)W^2(u)\mathrm{d}W(u)$, for $t \geq 0$.

b. Prove that the process $\{X(t) : t \geq 0\}$ is a martingale with respect to the filtration $\{\mathcal{F}(t) : t \geq 0\}$. Show that $\mathbb{E}(X(t)) = 1$ and $\mathbb{V}\mathrm{ar}[X(t)] = \int_0^t \mathbb{E}[W^4(u)X^2(u)]\mathrm{d}u$ for $t \geq 0$.

Z.O.Z. Remaining questions on the other side.

5. Given a Radon-Nikodym derivative Z, and the associated Radon-Nikodym process $\{Z(t) : t \geq 0\}$, defined by $Z(t) = \mathbb{E}[Z|\mathcal{F}(t)]$, where $\{\mathcal{F}(t) : t \geq 0\}$ is a given filtration. We then have the change of probability measure, $\mathrm{d}\tilde{\mathbb{P}} = Z\mathrm{d}\mathbb{P}$, with the expectation under the $\tilde{\mathbb{P}}$-measure, i.e., $\tilde{\mathbb{E}}[Y] = \mathbb{E}[ZY]$ Let Y be a random variable which is $\mathcal{F}(t)$-measurable. Prove (using partial averaging) that, for $s < t$

$$\tilde{\mathbb{E}}[Y|\mathcal{F}(s)] = \frac{1}{Z(s)}\mathbb{E}[YZ(t)|\mathcal{F}(s)].$$

6. Let $\{W(t) : 0 \leq t \leq T\}$ be a Brownian motion on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, and let $\{\mathcal{F}(t) : 0 \leq t \leq T\}$ be its natural filtration, and assume $\mathcal{F} = \mathcal{F}(T)$.
Consider a stock with price process $\{S(t) : 0 \leq t \leq T\}$ with

$$S(t) = S(0)\exp\left\{\int_0^t \mathrm{e}^{-u}\mathrm{d}W(u) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u})\mathrm{d}u\right\}$$

a. Let

$$X(t) = \int_0^t \mathrm{e}^{-u}\mathrm{d}W(u) + \int_0^t (1 - \frac{1}{2}\mathrm{e}^{-2u})\mathrm{d}u$$

Determine the distribution of $X(t)$.

b. Prove that $\{S(t) : t \geq 0\}$ is an Itô process.

Please, make sure that your name is written down on each of the submitted solutions.