Utrecht University
Mathematical Institute

Answers to Mid-Term Exam for Introduction to Financial Mathematics, WISB373

Monday December 12th 2022, 13:00 - 15:00 **(2 hours examination)**

1. Suppose a portfolio with a short position in the stock, i.e., $-S(t)$ plus a long position in a call option. Such a portfolio is called a cap.

Determine the payoff at time $t = T$ of the cap portfolio.

Derive an equivalent payoff based on a portfolio with a put option, and describe the instruments in this portfolio.

Answer: The payoff of having a short position and buying a call option is

$$-S_T + \max(S_T - K, 0) = \max(-S_T, -K) = -\min(S_T, K).$$

The payoff of having a purchased put combined with borrowing the discounted strike price at closing gives us the equivalent payoff:

$$\max(K - S_T, 0) - K = \max(-S_T, -K) = -\min(S_T, K),$$

which is the same as before.

By the put–call parity formula, both strategies have the same profit.

2. Let X and Y be two independent discrete random variables with distribution functions (CDFs) F_X and F_Y. Define

$$Z = \max(X, Y), \quad W = \min(X, Y).$$

Find the CDFs of Z and W.

Answer: To find the CDF of Z, we can write:

$$\begin{aligned}
F_Z(z) &= \mathbb{P}(Z \leq z) \\
&= \mathbb{P}(\max(X, Y) \leq z) \\
&= \mathbb{P}((X \leq z) \text{ and } (Y \leq z)) \\
&= \mathbb{P}((X \leq z)\mathbb{P}(Y \leq z) \quad \text{(since } X \text{ and } Y \text{ are independent)} \\
&= F_X(z)F_Y(z).
\end{aligned}$$

To find the CFD of W, we write:

$$\begin{aligned}
F_W(w) &= \mathbb{P}(W \leq w) \\
&= \mathbb{P}(\min(X, Y) \leq w) = 1 - \mathbb{P}(\min(X, Y) > w) \\
&= 1 - \mathbb{P}((X > w) \text{ and } (Y > w)) \\
&= 1 - \mathbb{P}((X > w)\mathbb{P}(Y > w) \quad \text{(since } X \text{ and } Y \text{ are independent)} \\
&= 1 - (1 - F_X(w))(1 - F_Y(w)) = F_X(w) + F_Y(w) - F_X(w)F_Y(w).
\end{aligned}$$

3. Let W be a Brownian motion. Show that $\{cW(t/c^2) : t \geq 0\}$ is a Brownian motion.

Answer: We need to check the conditions for $X(t) = cW(t/c^2)$ to be a Brownian motion.

1. $X_0 = cW_0 = 0$.
2. Since $X(t_k) - X(t_{k-1}) = cW(t_k/c^2) - cW(t_{k-1}/c^2)$ and $0 \leq t_0 < t_1 < \ldots t_k$, the random variables

$$W(t_1/c^2)-W(t_0/c^2), W(t_2/c^2)-W(t_1/c^2), \ldots W(t_k/c^2)-W(t_{k-1}/c^2),$$

are independent, which implies that

$$X(t_1) - X(t_0), X(t_2) - X(t_1), \ldots, X(t_k) - X(t_{k-1})$$

are independent.

3. Since cW is Gaussian if W is, $\forall s, t \geq 0$ such that $s < t$, $X_t - X_s = c\,(W(t/c^2) - W(s/c^2))$ is Gaussian with $\mathbb{E}[X(t) - X(s)] = c\mathbb{E}[W(t/c^2) - W(s/c^2)] = 0$ and variance $\mathbb{V}ar(X(t) - X(s)) = c^2\mathbb{V}ar(W(t/c^2) - W(s/c^2)) = c^2(t/c^2 - s/c^2) = t - s$.
4. $\forall \omega \in \Omega$, the path $t \to X(t)(\omega) = cW(t/c^2)(\omega)$ is continuous since $t \to W(t)(\omega)$ is.

4. A random variable Z with probability density function,

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},$$

is standard normally distributed, i.e. $Z \sim N(0,1)$, with $\mathbb{E}[Z] = 0, \mathbb{V}ar(Z) = 1$. For all $t \geq 0$, let $X(t) = \sqrt{t}Z$.

a. Determine $\mathbb{E}[|X(t)|]$ (i.e. the expected value of X absolute).

b. The stochastic process $X = \{X(t) : t \geq 0\}$ has continuous paths and $\forall t, X(t) \sim N(0,t)$. Is $X(t)$ a Brownian motion? Justify your answer.

Answer 4a:

Reasoning: Let $Z \sim N(0,1)$. Then, since $\sqrt{t}Z \sim N(0,t)$, we get

$$\begin{aligned}
\mathbb{E}[|X(t)|] &= \sqrt{t}\mathbb{E}[|Z|] \\
&= \sqrt{t}2 \int_0^\infty u \frac{e^{-u^2/2}}{\sqrt{2\pi}} du \\
&= \sqrt{t}2 \int_0^\infty \frac{e^{-w}}{\sqrt{2\pi}} dw \quad (w = u^2/2) \\
&= \frac{\sqrt{t}}{\sqrt{2\pi}} \left[-e^{-w}\right]_0^\infty = \sqrt{\frac{2t}{\pi}}
\end{aligned}$$

Answer 4b: No, since $0 \leq s \leq t < \infty$;

$$\begin{aligned}
\mathbb{V}\mathrm{ar}[X(t) - X(s)] &= [\sqrt{t}Z - \sqrt{s}Z] \\
&= \left(\sqrt{t} - \sqrt{s}\right)^2 [Z] \\
&= t - 2\sqrt{t}\sqrt{s} + s \neq t - s
\end{aligned}$$

Z.O.Z. Remaining questions on the other side.

5. Suppose $A_1, A_2, \ldots$ are independent random variables with mean zero and variance one and we write $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} A_i, \; n \geq 1.$$

Show that the proces $S_n - n$ is adapted to the filtration, and prove that the sequence $X_n = S_n^2 - n$ is a martingale.

Answer: $\mathbb{E}[S_n] = 0$ so $\mathbb{E}[S_n^2] = \mathbb{V}\mathrm{ar}[S_n] = \sum_{j=1}^{n} \mathbb{V}\mathrm{ar}[A_j] = n$ by the additivity of variance for sums of independent random variables.

But let us be careful to state that the fact that $\mathbb{E}[X_n] = X_0$ for all $n > 0$ is not by itself enough to imply that X_n is a martingale. In order to see whether the sequence is a martingale, we need to show that $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$. This requires us to put ourselves in the shoes of somebody who has all the information available up until stage n and to then work out what that somebody would consider the expectation of X_{n+1} to be. To this end, note that at time n, we know X_n and S_n, so a person with the information available at time n can treat X_n and S_n as known constants. The only new information that we get as time goes from n to $n+1$ is that we see the value A_{n+1}. Since we know nothing about A_{n+1}, its conditional mean and variance (given what we know up to stage n) are the same as its original mean and variance. So

$$\begin{aligned}
\mathbb{E}[X_{n+1}|\mathcal{F}_n] &= \mathbb{E}[S_{n+1}^2 - (n+1)|\mathcal{F}_n] \\
&= \mathbb{E}[(S_n^2 + A_{n+1})|\mathcal{F}_n] - (n+1) \\
&= \mathbb{E}[S_n^2 + 2A_{n+1}S_n + A_{n+1}^2] - (n+1) \\
&= S_n^2 + 0 + 1 - (n+1) = S_n^2 - n = X_n
\end{aligned}$$

6. Let X and Y be independent; each uniformly distributed on $[0,1]$. Let $Z = X + Y$. Find $\mathbb{E}[Z|X], \mathbb{E}[XZ|X]$ and $\mathbb{E}[XZ|Z]$ when it is known that $\mathbb{E}[X|Z] = Z/2$. Confirm your answer for $\mathbb{E}[Z|X]$ by making use of the iterated expectations property.

Answer:

$$\mathbb{E}[Z|X] = \mathbb{E}[X+Y|X] = E[X|X] + \mathbb{E}[Y|X] = X + \mathbb{E}[Y] = X + \frac{1}{2}$$

$$\mathbb{E}[XZ|Z] = Z\mathbb{E}[X|Z] = Z^2/2.$$

and

$$\mathbb{E}[XZ|X] = X\mathbb{E}[Z|X] = X(X + \frac{1}{2}).$$

Checks: We found $\mathbb{E}[Z|X] = X + 1/2$. So, its mean is $\mathbb{E}[X]| + 1/2 = 1/2 + 1/2 = 1$. Iterated expectations give us: $\mathbb{E}[\mathbb{E}[Z|X]] = \mathbb{E}[Z] = \mathbb{E}[X] + \mathbb{E}[Y] = 1/2 + 1/2 = 1$.

7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathbf{G} \subseteq \mathcal{F}$. Prove for $X = \mathbb{1}_B, B \in \mathbf{G}$, that if X is $\mathcal{G}$-measurable (so $\mathbb{E}[X|\mathbf{G}] = X$) then

$$\mathbb{E}[XY|\mathbf{G}] = X\mathbb{E}[Y|\mathbf{G}].$$

Answer: Proof: We show the RHS satisfies (i) and (ii):
Clearly, $X\mathbb{E}[Y|\mathbf{G}]$ is $\mathbf{G}$-measurable, since it is the product of two $\mathbf{G}$-measurable functions.
Step 1: $X = \mathbb{1}_B$, $B \in \mathbf{G}$ (since we want X to be $\mathbf{G}$-measurable). For any $A \in \mathbf{G}$,

$$\begin{aligned}\int_A X\mathbb{E}[Y|\mathbf{G}]\mathrm{d}\mathbb{P} &= \int_A \mathbb{1}_B\mathbb{E}[Y|\mathbf{G}]\mathrm{d}\mathbb{P} = \int_{A\cap B} \mathbb{E}[Y|\mathbf{G}]\mathrm{d}\mathbb{P} \\ &= \int_{A\cap B} Y\mathrm{d}\mathbb{P} = \int_A \mathbb{1}_B Y\mathrm{d}\mathbb{P} = \int_A XY\mathrm{d}\mathbb{P}.\end{aligned}$$

So the result holds for indicator functions.

Please, make sure that your name is written down on each of the submitted solution sheets.