Utrecht University
Mathematical Institute

Mid-Term Exam for Introduction to Financial Mathematics, WISB373

Monday December 12th 2022, 13:00 - 15:00 **(2 hours examination)**

For each of the exercises 10 points can be obtained.

1. Suppose a portfolio with a short position in the stock, i.e., $-S(t)$ plus a long position in a call option. Such a portfolio is called a cap.

Determine the payoff at time $t = T$ of the cap portfolio.

Derive an equivalent payoff based on a portfolio with a put option, and describe the instruments in this portfolio.

2. Let X and Y be two independent discrete random variables with distribution functions (CDFs) F_X and F_Y. Define

$$Z = \max(X, Y), \quad W = \min(X, Y).$$

Find the CDFs of Z and W.

3. Let W be a Brownian motion. Show that $\{cW(t/c^2) : t \geq 0\}$ is a Brownian motion.

4. A random variable Z with probability density function,

$$f(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}},$$

is standard normally distributed, i.e. $Z \sim N(0, 1)$, with $\mathbb{E}[Z] = 0, \mathbb{V}\mathrm{ar}(Z) = 1$. For all $t \geq 0$, let $X(t) = \sqrt{t}Z$.

a. Determine $\mathbb{E}[|X(t)|]$ (i.e. the expected value of X absolute).

b. The stochastic process $X = \{X(t) : t \geq 0\}$ has continuous paths and $\forall t, X(t) \sim N(0, t)$. Is $X(t)$ a Brownian motion? Justify your answer.

Z.O.Z. Remaining questions on the other side.

5. Suppose $A_1, A_2, \ldots$ are independent random variables with mean zero and variance one and we write $S_0 = 0$ and

$$S_n = \sum_{i=1}^{n} A_i, \; n \geq 1.$$

Show that the proces $S_n - n$ is adapted to the filtration, and prove that the sequence $X_n = S_n^2 - n$ is a martingale.

6. Let X and Y be independent; each uniformly distributed on $[0,1]$. Let $Z = X + Y$. Find $\mathbb{E}[Z|X], \mathbb{E}[XZ|X]$ and $\mathbb{E}[XZ|Z]$ when it is known that $\mathbb{E}[X|Z] = Z/2$. Confirm your answer for $\mathbb{E}[Z|X]$ by making use of the iterated expectations property.

7. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{G} \subseteq \mathcal{F}$. Prove for $X = \mathbb{1}_B, B \in \mathcal{G}$, that if X is $\mathcal{G}$-measurable (so $\mathbb{E}[X|\mathcal{G}] = X$) then

$$\mathbb{E}[XY|\mathcal{G}] = X\mathbb{E}[Y|\mathcal{G}].$$

Please, make sure that your name is written down on each of the submitted solution sheets.