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Stochastic Processes: Retake, 2022-23 You are allowed to use the *Summary of Important Results Parts 1 and 2*

(1) Consider the Markov chain $(X_n)_{n\geq 0}$ with state space $I = \{1,2,3,4\}$ and transition matrix

$$\begin{bmatrix} \frac{1}{3} & \frac{2}{3} & 0 & 0 \\ \frac{1}{4} & 0 & 0 & \frac{3}{4} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

  (a) Determine the communicating classes, which ones are recurrent and which ones are transient. Justify your answer. (1 pt)

  (b) For $i = 1,2,3,4$, determine the distribution of T_i under the conditional measure $\mathbb{P}_i$, where $T_i = \inf\{n \geq 1 : X_n = i\}$ is the first passage time to state i. (Hint: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$, for $|x| < 1$.) (1.5 pts)

  (c) Determine $\mathbb{E}_i[T_i]$ for $i = 1,2,3,4$. (Hint: $\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}$, for $|x| < 1$.) (1 pt)

(2) Computers produced at a certain factory undergo regular quality checks. There are three kinds of tests performed: type 1 (heavy check), type 2 (medium check) and type 3 (light check). For each type test there are two possible outcomes: pass or fail. Let $0 < p_i < 1$ be the probability that a computer passes type i test, $i = 1,2,3$. We assume that the result of any test performed does not depend on the results of the previous tests. If a computer passes a test (regardless of which test is performed), then with equal probability a test is chosen for the next computer. If on the other hand a computer fails the performed test, then the next computer undergoes a type 1 test. Let X_n be the type of the n-th test.

  (a) Argue why $(X_n)_{n\geq 0}$ is a Markov chain and determine its transition matrix. (1 pt)

  (b) Determine $\lim_{n\to\infty} \mathbb{P}(X_n = j)$ for $j = 1,2,3$. (1 pt)

  (c) Consider $T_1 = \inf\{n \geq 1 : X_n = 1\}$, the first passage time to state 1. Determine the value of $\mathbb{P}(X_{T_1+2} = 2)$. (1 pt)

(3) James receives e-mails starting at 10 am according to a Poisson process with rate 10 e-mails per hour.

  (a) Find the probability that he will receive exactly 18 e-mails by noon and 70 e-mails by 5 pm? (1 pt)

  (b) Suppose James received 40 e-mails by 3 pm, what is the probability that he has received 20 e-mails by 1 pm? (1 pt)

(4) Shocks occur according to a Poisson process with rate λ, and each shock independently causes a certain system failure with probability p. Let X_t be the number of shocks in the interval $[0,t]$. Denote by T the time at which the system fails and let N be the number of shocks until the system fails. Note that N is independent of the times that the shocks take place. Prove that T is exponentially distributed with parameter $p\lambda$. (1.5 pts)