MANIFOLDS(WISB342) EXAM (NOVEMBER 8, 2022)

Exercise 1 (1pt)**.** You know already, from one of the homeworks, that $\mathbb{P}^2$ can be embedded in $\mathbb{R}^4$. Show now that $\mathbb{P}^2$ can be embedded in S^4. (Hint: do not look for complicated formulas).

Exercise 2 (2pt)**.** Let $f : M \to N$ be a smooth map between two manifolds of dimensions m and n, respectively, let $N_0 \subset N$ be a (smooth, embedded) submanifold and we are interested in the pre-image

$$M_0 = f^{-1}(N_0) := \{x \in M : f(x) \in N_0\}.$$

The RVT (regular value theorem) tells us that if N_0 consists of a single point and we require f to be a submersion at all points in M_0, then M_0 is a submanifold of M of dimension $n - m$, whose tangent spaces are given by the kernels of the differentials of f:

$$T_x M_0 = \{v \in T_x M : (df)_x(v) = 0\} \quad (\text{for } x \in M_0).$$

Here we want to generalise the RVT to more general submanifolds N_0. To that end, we replace the submersion condition by the condition that "f is transverse to N_0" by which we mean: for each $x \in M_0$ one has $(df)_x(T_x M) + T_{f(x)}N_0 = T_{f(x)}N$ or, more explicitly: any element $w \in T_{f(x)}N$ can be written as

$$w = (df)_x(v) + w_0, \quad \text{with } v \in T_x M, w_0 \in T_{f(x)}N_0.$$

a) Further assuming that $N_0 = g^{-1}(z)$ for some submersion $g : N \to P$ and $z \in P$ into yet another manifold P, show that, indeed, M_0 is an submanifold of M.
b) Describe the dimension of M_0 in terms of the dimensions of M, N and N_0.
c) Describe the tangent spaces of M_0 in terms of the ones of M, N, N_0 and the differential of f.
d) Finally, show that the conclusions above hold in general (without assuming g).

Exercise 3 (7pts)**.** Consider the following curve in $\mathbb{R}^3$:

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \quad \gamma(t) = (t^2, t^3, t).$$

a) Show that the following is a submanifold of $\mathbb{R}^3$ containing γ:

$$M = \{(x, y, z) \in \mathbb{R}^3 : y = xz\}.$$

b) Show that the following defines a vector field on M

$$V := 2z\frac{\partial}{\partial x} + (x + 2z^2)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

c) Show that γ is an integral curve of V.
d) Here are three more subspaces of $\mathbb{R}^3$ that contain γ:

$$\{(x, y, z) \in \mathbb{R}^3 : x = z^2\}, \quad \{(x, y, z) \in \mathbb{R}^3 : x^3 = y^2\}, \quad \{(x, y, z) \in \mathbb{R}^3 : y = z^3\}.$$

Among them, only one is not a submanifold of $\mathbb{R}^3$. Which one, and why is it not?
e) Find vector fields $X, Y \in \mathfrak{X}(M)$ which, at each point in M, give a basis of the tangent space of M.
f) Compute $[X, Y]$.
g) $^{1\text{pt}}$ Compute the flow of V, describing explicitly all the diffeomorphisms ϕ_V^t induced by V.
h) Find a non-zero 1-form $\theta \in \Omega^1(\mathbb{R}^3)$ such that $\theta|_M = 0$.
i) Find a 1-form on M which is not exact, i.e. cannot be written as df for some $f \in C^\infty(M)$.
j) Show that $\mu = (dx \wedge dz)|_M$ is a volume form.
k) For $\omega = e^y(x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy)|_M \in \Omega^2(M)$ find $f \in C^\infty(M)$ such that $\omega = f \cdot \mu$.
l) Compute $i_V(\omega)$.
m) Also compute $L_V(\omega)$, writing the result in the form $g \cdot \mu$ (with g a function explicitly computed).

Exercise 4 (1pt)**.** Let's wonder whether a manifold M can admit a volume form μ of type $\mu = \theta \wedge \theta$, for some other differential form θ on M. Show that if this happens then the dimension of M is divisible by 4. Then show that this can happen already on $M = \mathbb{R}^4$ (provide an explicit example!).

NOTES:

- Please write down your name CLEARLY (this is important for me to be able to give you feedback via Microsoft Teams), and do not forget to mention also the student no!
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! In particular, please include all your computations that support your claims.
- Each subquestion (labelled by a letter) is worth 0.5 points, except for g) of Exercise 3 (1pt, if you provide all details, as mentioned above).
- The mark for the exam is the minimum between 10 and the total number of points you collect.
- The order of the exercises is completely unrelated to their difficulty (in particular, there is absolutely no reason to be scared about the first or the last exercise!).