MANIFOLDS(WISB342) RETAKE EXAM (DECEMBER 20, 2022)

Exercise 1 (1pt). Prove that $N = \{(x,y) \in \mathbb{R}^2 : x^3 = y^2\}$ is not an embedded submanifold of $\mathbb{R}^2$.

Exercise 2 (2pt). Let $f : M \to N$ be a smooth map between two manifolds of dimensions m and n, respectively, let $N_0 \subset N$ be a (smooth, embedded) submanifold and we are interested in the pre-image

$$M_0 = f^{-1}(N_0) := \{x \in M : f(x) \in N_0\}.$$

The RVT (regular value theorem) tells us that if N_0 consists of a single point and we require f to be a submersion at all points in M_0, then M_0 is a submanifold of M of dimension $n - m$, whose tangent spaces are given by the kernels of the differentials of f:

$$T_x M_0 = \{v \in T_x M : (df)_x(v) = 0\} \quad (\text{for } x \in M_0).$$

Here we want to generalise the RVT to more general submanifolds N_0. To that end, we replace the submersion condition by the condition that "f is transverse to N_0" by which we mean: for each $x \in M_0$ one has $(df)_x(T_x M) + T_{f(x)}N_0 = T_{f(x)}N$ or, more explicitly: any element $w \in T_{f(x)}N$ can be written as

$$w = (df)_x(v) + w_0, \quad \text{with } v \in T_x M, w_0 \in T_{f(x)}N_0.$$

a) Further assuming that $N_0 = g^{-1}(z)$ for some submersion $g : N \to P$ and $z \in P$ into yet another manifold P, show that, indeed, M_0 is an submanifold of M.
b) Describe the dimension of M_0 in terms of the dimensions of M, N and N_0.
c) Describe the tangent spaces of M_0 in terms of the ones of M, N, N_0 and the differential of f.
d) Finally, show that the conclusions above hold in general (without assuming g).

Exercise 3 (6.5pts). Consider the following curve in $\mathbb{R}^3$:

$$\gamma : \mathbb{R} \to \mathbb{R}^3, \quad \gamma(t) = (t^2, t^3, t).$$

a) Show that the following is a submanifold of $\mathbb{R}^3$ containing γ:

$$M = \{(x,y,z) \in \mathbb{R}^3 : y = xz\}.$$

b) Show that the following defines a vector field on M

$$V := 2z\frac{\partial}{\partial x} + (x + 2z^2)\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

c) Show that γ is an integral curve of V.
e) Find vector fields $X, Y \in \mathfrak{X}(M)$ which, at each point in M, give a basis of the tangent space of M.
f) Compute $[X, Y]$.
g) 1pt Compute the flow of V, describing explicitly all the diffeomorphisms ϕ_V^t induced by V.
h) Find a non-zero 1-form $\theta \in \Omega^1(\mathbb{R}^3)$ such that $\theta|_M = 0$.
i) Find a 1-form on M which is not exact, i.e. cannot be written as df for some $f \in C^\infty(M)$.
j) Show that $\mu = (dx \wedge dz)|_M$ is a volume form.
k) For $\omega = e^y(x \cdot dy \wedge dz + y \cdot dz \wedge dx + z \cdot dx \wedge dy)|_M \in \Omega^2(M)$ find $f \in C^\infty(M)$ such that $\omega = f \cdot \mu$.
l) Compute $i_V(\omega)$.
m) Also compute $L_V(\omega)$, writing the result in the form $g \cdot \mu$ (with g a function explicitly computed).

Exercise 4 (1pt). Consider the following copy of the torus in $\mathbb{R}^3$:

$$\mathbb{T}^2 := \{(x,y,z) \in \mathbb{R}^3 : \left(\sqrt{x^2+y^2} - 2\right)^2 + z^2 = 1\}.$$

Choose an orientation on $\mathbb{T}^2$ and compute

$$\int_{\mathbb{T}^2} \omega, \quad \text{where } \omega = (x^{2022}y^{2023}dx \wedge dy)|_{\mathbb{T}^2}.$$

NOTES:

- Please write down your name CLEARLY (this is important for me to be able to give you feedback via Microsoft Teams), and do not forget to mention also the student no!
- PLEASE MOTIVATE ALL YOUR ANSWERS!!!! In particular, please include all your computations that support your claims.
- Each subquestion (labelled by a letter) is worth 0.5 points, except for g) of Exercise 3 (1pt, if you provide all details, as mentioned above).
- The mark for the exam is the minimum between 10 and the total number of points you collect.
- The order of the exercises is completely unrelated to their difficulty.