Student number: .... Name: ....

Final exam

Topologie en Meetkunde, Block 3, 2023

Instructions

- The exam is closed-book.
- Write your name and student number in all the pages of the exam.
- You may write your solutions in either Dutch or English.
- You must justify the claims you make.
- You may use results from the lectures or the dictaat, but you must provide a clear statement (with complete hypothesis and conclusion).
- Try to write with clear handwriting. Structure your explanations clearly, using one paragraph for each new idea and one sentence for each particular claim.
- Advice: Read all the exam in the beginning and address first the questions that you find easier.

Questions

Exercise 1 (1 point)**.** Let $A \subset B$ be a retract.

- Suppose that B is contractible. Prove that A is also contractible.
- Find an example in which A is contractible but B is not.

Exercise 2 (1 point)**.** Fix a set S. We let $\mathcal{C}$ be its pair groupoid $S \times S \rightrightarrows S$. Given elements $x, y \in S = \mathrm{Ob}(\mathcal{C})$:

- Do they have a product in the category $\mathcal{C}$? If so, determine it.
- Do they have a coproduct in the category $\mathcal{C}$? If so, determine it.

Exercise 3 (2.5 points)**.** Let $f : \mathbb{S}^1 \to \mathbb{S}^2$ be the inclusion of the equator and $g : \mathbb{S}^1 \to \mathbb{RP}^2$ a constant map. Consider the space

$$X = \mathrm{pushout}(\mathbb{S}^2 \xleftarrow{f} \mathbb{S}^1 \xrightarrow{g} \mathbb{RP}^2).$$

- Endow X with the structure of a 2-dimensional cell-complex.
- Compute the fundamental group of X.

- Compute the first homologies of X.
- Is X a surface?

Exercise 4 (1 point)**.** Let (A, a) be a pointed space with fundamental group isomorphic to

$$G = \langle g_1, \ldots, g_n \mid r_1, \cdots, r_m \rangle.$$

Find a pointed space (B, b) and an inclusion $i : (A, a) \to (B, b)$ such that:

- $\pi_1(B, b)$ is isomorphic to the abelianisation of $\pi_1(A, a)$,
- $i_* : \pi_1(A, a) \longrightarrow \pi_1(B, b)$ is precisely the canonical homomorphism from $\pi_1(A, a)$ to its abelianisation.

Exercise 5 (1 point)**.** Construct a 2-dimensional cell complex whose fundamental group is not isomorphic to the fundamental group of a closed surface.

Exercise 6 (1 point)**.** Let Y be a path-connected manifold with finite fundamental group. Prove that every map $Y \to \mathbb{S}^1$ is nullhomotopic.

Exercise 7 (2.5 points)**.** Let $(A, a) = (\mathbb{RP}^2, q) \vee (\mathbb{RP}^2, q)$.

- Produce a 2-sheeted covering map $\pi : (B, b) \to (A, a)$, with B path-connected. You should justify that π is indeed covering.
- Compute the fundamental group of (B, b). Compute its image $\mathrm{im}(\pi_*) \subset \pi_1(A, a)$.
- Let X be the space introduced in Exercise 3. Is X a covering space of A?