Student number: .... Name: ....

Retake

Topologie en Meetkunde, Block 3, 2023

Instructions

- The exam is closed-book.
- Write your name and student number in all the pages of the exam.
- You may write your solutions in either Dutch or English.
- You must justify the claims you make.
- You may use results from the lectures or the dictaat, but you must provide a clear statement (with complete hypothesis and conclusion).
- Try to write with clear handwriting. Structure your explanations clearly, using one paragraph for each new idea and one sentence for each particular claim.
- Advice: Read all the exam in the beginning and address first the questions that you find easier.

Questions

Exercise 1 (1 point)**.** Find a space A and a subspace $B \subset A$ such that:

- B is a retract of A,
- B is not contractible,
- B is not a deformation retract of A.

Exercise 2 (1.5 points)**.** Let X be a space. We define its category of opens $\mathcal{O}(X)$ as follows:

- Objects in $\mathcal{O}(X)$ are open subsets $U \subset X$.
- For each pair of objects $U \subset V$ in $\mathcal{O}(X)$, $\mathrm{Hom}(U, V)$ contains a single element, the inclusion $i_{UV} : U \to V$. Otherwise, if U is not a subset of V, $\mathrm{Hom}(U, V)$ is empty.

Then:

- Verify that $\mathcal{O}(X)$ is a category.
- Prove that $U \cap V$ is the product of U and V, as elements of $\mathcal{O}(X)$.

- Prove that $U \cup V$ is the coproduct of U and V, as elements of $\mathcal{O}(X)$.

Exercise 3 (1 point)**.** Let $\gamma_k : \mathbb{S}^1 \to \mathbb{S}^1$ be the map $\gamma_k(z) = z^k$. Prove that the pushforward

$$(\gamma_k)_* : \Pi_1(\mathbb{S}^1) \to \Pi_1(\mathbb{S}^1)$$

is a groupoid isomorphism if and only if $k = \pm 1$.

Exercise 4 (1 point)**.** Construct a 2-dimensional cell complex, homotopy equivalent to the 2-torus, but which is not a surface.

Exercise 5 (2 points)**.** Let $f : \mathbb{S}^1 \to \mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$ be the map $z \mapsto (z, z^2)$ and $g : \mathbb{S}^1 \to \mathbb{S}^2$ the inclusion of the equator. Consider the space

$$X := \text{pushout}(\mathbb{T}^2 \xleftarrow{f} \mathbb{S}^1 \xrightarrow{g} \mathbb{S}^2).$$

- Endow X with the structure of a 2-dimensional cell-complex.
- Compute the fundamental group of X.
- Compute the first homologies of X.

Exercise 6 (3.5 points)**.** Consider the space $A := \mathbb{RP}^2 \vee \mathbb{T}^2$. Fix a basepoint a.

- Endow A with a cell structure.
- Compute the fundamental group of (A, a).
- Produce a 2-sheeted covering map $\pi : (B, b) \to (A, a)$, with B path-connected.
- Compute the fundamental group of (B, b). Compute its image $\text{im}(\pi_*) \subset \pi_1(A, a)$.
- Produce a 2-sheeted covering map $\tau : (C, c) \to (A, a)$, not isomorphic to π, with C path-connected.

You have to justify that π and τ are indeed covering.