WISB333 2022 Examination Problems

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1 Hopf bifurcation in ZH normal form

Consider the planar system

$$\begin{cases} \dot{\xi} &= \beta_1 + \xi^2 + \rho^2, \\ \dot{\rho} &= \rho(\beta_2 + \theta\xi + \xi^2), \end{cases} \tag{1.1}$$

where $\theta < 0$. This system appears in the analysis of the fold-Hopf codim 2 bifurcation of equilibria as an amplitude system for the truncated normal form, so that $\rho \geq 0$.

1. Find parameter values at which bifurcations of equilibria with $\rho = 0$ occur.

2. Verify that Hopf bifurcation of an equilibrium with small $\rho > 0$ happens in the system (1.1) at the line

   $$T = \{(\beta_1, \beta_2) : \beta_2 = 0,\ \beta_1 < 0\}.$$

   Derive an expression for the first Lyapunov coefficient l_1 along the Hopf line T and predict stability of the bifurcating cycle.

3. Illustrate your predictions by simulations with `pplane` and by numerical continuation in `MatCont`.

4. Try to obtain as complete as possible bifurcation diagram of (1.1) near the origin for small $\|\beta\|$.

2 Hopf bifurcation in R2 normal form

Consider the planar system

$$\begin{cases} \dot{\zeta}_1 &= \zeta_2, \\ \dot{\zeta}_2 &= \beta_1\zeta_1 + \beta_2\zeta_2 - \zeta_1^3 - \zeta_1^2\zeta_2. \end{cases} \tag{2.1}$$

This system appears in the study of codim 2 bifurcation of limit cycles corresponding to a double multiplier -1.

1. Verify that Hopf bifurcation of the trivial equilibrium $(\zeta_1, \zeta_2) = (0, 0)$ of (2.1) happens at the line
$$H^{(1)} = \{(\beta_1, \beta_2) : \beta_2 = 0,\ \beta_1 < 0\},$$
2. Verify that Hopf bifurcation of the nontrivial equilibria $(\zeta_1, \zeta_2) \neq (0, 0)$ of (2.1) happens at the line
$$H^{(2)} = \{(\beta_1, \beta_2) : \beta_1 = \beta_2,\ \beta_1 > 0\},$$
3. Compute symbolically the first Lyapunov coefficient l_1 along the lines $H^{(1)}$ and $H^{(2)}$ and predict stability of the bifurcating cycles.
4. Verify your results by simulations with `pplane` and by numerical continuation in `MatCont`.
5. Try to obtain as complete as possible bifurcation diagram of (2.1) near the origin for small $\|\beta\|$.

3 Hopf bifurcation in Selkov's model

Consider the following simplifyed model of glycolysis

$$\begin{cases} \dot{x} &= -x + ay + x^2 y, \\ \dot{y} &= b - ay - x^2 y \end{cases} \tag{3.1}$$

where x is the concentration ADP (adenosine diphosphate) and y is the concentration F6P (fructose-6-phosphate). The parameters a, b affect the speed of the reaction from F6P to ADP and the addition of F6P, respectively.

1. Show that the triangle in the positive quadrant bounded by $y + x \leq C$ contains a trapping region for some constant C. Pay attention to the points $(x, y) = (b, b/a)$ and the point where the triangle hits the x-nullcline.

2. Determine a curve $a(b)$ on which the equilibrium exhibits a Hopf bifurcation and compute the first Lyapunov coefficient.

3. Use Poincaré-Bendixson to classify solutions of the system.

4. Illustrate your predictions by simulations in `pplane`.

4 Hopf bifurcation in the advertising model by Feichtinger

Consider the following planar system

$$\begin{cases} \dot{x}_1 &= \alpha[1 - x_1 x_2^2 + A(x_2 - 1)], \\ \dot{x}_2 &= x_1 x_2^2 - x_2, \end{cases} \tag{4.1}$$

where $A > 0$ is constant and α is a bifurcation parameter.

1. Check that the system (4.1) has an equilibrium that exhibits a Hopf bifurcation at some value $\alpha_{\mathrm{H}} = \alpha_{\mathrm{H}}(A)$ of parameter α. Verify also that the eigenvalues cross the imaginary axis at nonzero velocity w.r.t. the parameter.

2. Compute the corresponding first Lyapunov coefficient l_1 symbolically.

3. Confirm your results by simulations with `pplane` or by numerical continuation of the limit cycle in `MatCont`.

5 Adaptive control - I

Consider the following system from Control Theory

$$\begin{cases} \dot{x} &= 1 + x - xy, \\ \dot{y} &= \alpha y + \beta x^2, \end{cases} \tag{5.1}$$

where (α, β) are parameters. Assume that $\alpha < 0$.

(a) Find fold and Hopf bifurcation curves of (5.1) in the parameter half-plane $\alpha < 0$.

(b) Verify that (5.1) exhibits a Bogdanov-Takens (BT) bifurcation at $(\alpha, \beta) = (-\frac{2}{3}, \frac{8}{81})$, i.e., has an equilibrium with a double zero eigenvalue. (*Hint*: This is a common point with $\alpha < 0$ of the fold and Hopf bifurcation curves.)

(c) Compute the coefficients (a, b) of the critical BT normal form and prove that the corresponding bifurcation is nondegenerate, i.e.,

$$ab \neq 0.$$

(d) Fix $\alpha = -0.25$ and produce with `pplane` the phase portraits for $\beta = 0.05, 0.03, 0.01,$ and 0.003.

(e) Sketch the bifurcation diagram of (5.1). Where do you put the homoclinic bifurcation curve?

(f) Indicate the region where the system has a stable equilibrium.

6 Averaged forced Van der Pol oscillator

Consider the following planar system studied by Holmes and Rand:

$$\begin{cases} \dot{x} &= -\omega y + x(1 - x^2 - y^2), \\ \dot{y} &= \omega x + y(1 - x^2 - y^2) - F, \end{cases} \tag{6.1}$$

where (ω, F) are positive parameters.

(a) Find fold and Hopf bifurcation curves of (6.1) in the first quadrant of the parameter plane, i.e. when $\omega, F > 0$. (*Hint*: In both cases, express F as functions of ω.)

(b) Verify that two branches of the fold curve in (6.1) meet at a cusp point (CP) with

$$\omega = \left(\frac{1}{3}\right)^{1/2}, \quad F = \left(\frac{2}{3}\right)^{3/2}.$$

(c) Verify that (6.1) exhibits a Bogdanov-Takens (BT) bifurcation at

$$\omega = F = \frac{1}{2},$$

i.e., has an equilibrium with a double zero eigenvalue.

(d) Compute the coefficients (a, b) of the critical BT normal form and prove that the corresponding bifurcation is nondegenerate, i.e.,

$$ab \neq 0.$$

(e) Sketch the bifurcation diagram of (6.1). Where do you put the homoclinic bifurcation curve?

(f) Verify your results by simulations with `pplane` and by numerical continuation in `MatCont`.

7 Prey-predator dynamics - I

Consider the following prey-predator model depending on two positive parameters (α, δ)

$$\begin{cases} \dot{x} &= x - \dfrac{xy}{1+\alpha x}, \\ \dot{y} &= -y - \delta y^2 + \dfrac{xy}{1+\alpha x}, \end{cases} \tag{7.1}$$

for $x, y \geq 0$.

1. Derive equations for the saddle-node and Hopf bifurcations of positive equilibria in the system. *Hint*: Consider the orbitally-equivalent to (7.1) polynomial system

$$\begin{cases} \dot{x} &= x(1+\alpha x) - xy, \\ \dot{y} &= -(y+\delta y^2)(1+\alpha x) + xy. \end{cases} \tag{7.2}$$

2. Prove that a Bogdanov-Takens bifurcation occurs in the system (7.2) and find the corresponding parameter values.

3. Compute the coefficients a and b of the BT-normal form.

4. Use `pplane` and `MatCont` to produce representative phase portraits of the model and to sketch its simplest possible bifurcation diagram.

8 Prey-predator dynamics - II

Consder the following prey-predator model depending on two parameters (l, m)

$$\begin{cases} \dot{x} &= x(x-l)(1-x) - xy, \\ \dot{y} &= -y(m-x), \end{cases} \tag{8.1}$$

where $m > 0$ and $0 < l < 1$, and $x, y \geq 0$.

1. Derive equations for the borders of a domain in the (l, m)-plane, in which the model has a positive equilibrium.

2. Derive an equation for the Hopf bifurcation of the positive equilibrium of (8.1). Prove that this bifurcation is supercritical, i.e. gives rise to a stable periodic orbit.

3. Use `pplane` and `MatCont` to produce representative phase portraits of the model and sketch its simplest possible bifurcation diagram. *Hints*: Fix $l = \frac{1}{2}$ and plot the phase portraits for several different values of m.

4. There is a global (heteroclinic) bifurcation in the system. Find numerically m_{HET} for the heteroclinic parameter value when $l = \frac{1}{2}$.

9 Adaptive control - II

Consider the following 3D system

$$\begin{cases} \dot{x} &= \mu x + y, \\ \dot{y} &= -x + \mu y - xz, \\ \dot{z} &= -z + ax^2, \end{cases} \tag{9.1}$$

where $a \neq 0$. This system appears in the control theory.

1. Verify that system (9.1) exhibits a Hopf bifurcation of the equilibrium $(x, y, z) = (0, 0, 0)$ at the parameter value $\mu = 0$.

2. Compute the corresponding first Lyapunov coefficient l_1 and predict the direction of the Hopf bifurcation and the stability of the bifurcating limit cycle.

3. Verify your predictions by simulations or by numerical continuation of the cycle in `MatCont`.

10 Hopf bifurcation in the Coimbrator

Consider the following model for changes of atmospheric CO_2 that includes photosynthesis and the carbon cycle:

$$\begin{cases} \dot{x} &= \alpha x - xy, \\ \dot{y} &= -y - xy + 2\beta z, \\ \dot{z} &= \alpha x - \beta z, \end{cases} \tag{10.1}$$

where α, β are positive parameters.

1. Investigate the linear stability of both equilibria.
2. Find a Hopf bifurcation curve of one equilibrium, i.e., determine the corresponding $\alpha = \alpha_{\mathrm{H}}(\beta)$.
3. Compute the first Lyapunov coefficient l_1.
4. Illustrate your findings using simulations in `MatCont`.