Retake exam WISB326, July 10, 2023, 09:00-12:00

Exam problems

- Let k be an algebraically closed field of characteristic 0.

(1) Let $X = V(x^3 - x^2y + 2xyz + xz^2 - 2y^2z - yz^2) \subseteq \mathbb{P}^2(k)$. Let $Z = X \setminus V(x - y)$.
  (a) (6 points) Determine the irreducible components of X.
  (b) (4 points) Compute $I(Z)$.

(2) (8 points) Let $I \subseteq k[x_0, \dots, x_n]$ be a homogeneous ideal. Show that I is radical if and only if every homogeneous polynomial $f \in k[x_0, \dots, x_n]$ such that $f^n \in I$ for some $n \geq 1$ satisfies $f \in I$.

(3) (4 points) For every $a = (a_{0,0}, a_{0,1}, a_{0,2}, a_{1,1}, a_{1,2}, a_{2,2}) \in k^6$, let $f_a = \sum_{0 \leq i \leq j \leq 2} a_{i,j} x_i x_j$. Let X be the set of points $a \in \mathbb{A}^6(k)$ such that $V(f_a) \subseteq \mathbb{P}^2(k)$ contains the point $(0 : 0 : 1)$. Let U be the set of points in X such that f_a is a projective plane curve and $I((0 : 0 : 1), f_a \cap x_0) = 1$. Show that X is an algebraic set in $\mathbb{A}^6(k)$. Show that U is an open subset of X.

(4) Let $f \in k[x, y, z]$ be a homogeneous polynomial of degree $d \geq 1$.
  (a) (3 points) Show that if the projective plane curve $V(f)$ is nonsingular, then f is an irreducible polynomial.
  (b) (4 points) Let $L \subseteq \mathbb{P}^2(k)$ be a line. Show that if f is irreducible and there are distinct points $P, Q \in L \cap V(f)$ such that P, Q are singular for f and L is tangent to $V(f)$ at both P and Q, then $d \geq 6$.

(5) Consider the projective plane curve $X = V(f)$ given by

$$f = 2x_0^5 + (x_0x_1^2 - 4x_1^3)(4x_1 + x_2)^2.$$

  (a) (2 points) Compute a change of coordinates $\varphi : \mathbb{P}^2(k) \to \mathbb{P}^2(k)$ such that $\varphi(0 : 0 : 1) = (0 : 1 : -4)$, $\varphi(0 : 1 : 0) = (0 : 0 : 1)$ and $\varphi(V(x_0)) = V(x_0)$.
  (b) (3 points) Compute the intersection number $I((0 : 0 : 1), f \cap (x_0 - 4x_1))$.
  (c) (16 points) Compute the multiple points for f. Compute multiplicites, tangent lines and multiplicities of the tangent lines at each multiple point for f.

(6) Let X and Y be two varieties. Let $\varphi : X \to Y$ be a morphism. Let $Z = \overline{\varphi(X)}$ be the Zariski closure of $\varphi(X)$ in Y.
  (a) (6 points) Show that Z is a variety.
  (b) (4 points) Show that $\dim(Z) \leq \dim(X)$.

(7) Consider the morphism

$$\varphi : \mathbb{A}^2(k) \to \mathbb{A}^2(k), \quad (x, y) \mapsto (x, xy).$$

(a) (3 points) Show that φ is birational and find an explicit inverse map ψ.

(b) (4 points) Let $X = V(y^2 - x^2(x-1)) \subseteq \mathbb{A}^2(k)$. Let $C = \varphi^{-1}(X)$. Show that C is the union of two nonsingular irreducible affine plane curves C_1 and C_2.

(c) (4 points) For $i \in \{1, 2\}$, determine whether C_i is birationally equivalent to X.

(d) (9 points) Show that ψ is not a morphism and compute the domain of ψ.