Measure and Integration: Final Exam 2022-23
You are allowed a four two-sided A4s with any information you want

(1) Consider the measure space $((0,\infty),\mathcal{B}((0,\infty),\lambda)$, where $\mathcal{B}((0,\infty))$ and λ are the restrictions of the Borel σ-algebra and Lebesgue measure to the interval $(0,\infty)$. Show that

$$\lim_{n\to\infty}\int_{(0,n)}\left(1+\frac{x}{n}\right)^n e^{-4x}\,d\lambda(x) = 1/3.$$

(Hint: note that $1+x \le e^x$ and $\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = e^x$). (1.5 pts)

(2) Let $(X,\mathcal{A},\mu)$ be a measure space and $u_n, v_n, u, v \in \mathcal{L}^1(\mu)$ for $n \ge 1$. Assume that
(i) $\lim_{n\to\infty} u_n = u$. μ a.e. and $\lim_{n\to\infty} v_n = v$. μ a.e.
(ii) $v_n \ge 0$ and $|u_n| \le v_n$ for $n \ge 1$.
(iii) $\lim_{n\to\infty}\int v_n d\mu = \int v d\mu$.

(a) Prove that $\int u d\mu \le \liminf_{n\to\infty}\int u_n d\mu$. (Hint: apply Fatou's Lemma to the sequence $(u_n+v_n)_n$, justify first that you can use the lemma) (1.5 pts)

(b) Prove that $\int u d\mu \ge \limsup_{n\to\infty}\int u_n d\mu$, and conclude that $\int u d\mu = \lim_{n\to\infty}\int u_n d\mu$. (Hint: apply Fatou's Lemma to the sequence $(v_n-u_n)_n$, justify first that you can use the lemma) (1.5 pts)

(3) Let $(X,\mathcal{A},\mu)$ be a measure space and $u \in \mathcal{L}^2(\mu)\bigcap\mathcal{L}^\infty(\mu)$. Set $A = \{x \in X : |u(x)| \ge 1\}$.

(a) Prove that $\mu(A) < \infty$. (0.5 pt)

(b) Prove that $\mathbb{I}_A u \in \mathcal{L}^1(\mu)$ and that $\|\mathbb{I}_A u\|_1 \le (\mu(A))^{\frac{1}{2}}\|u\|_2$. (0.5 pt)

(c) Prove that for all $p \in [2,\infty)$, $u \in \mathcal{L}^p(\mu)$ and $\|u\|_p \le \left(\|u\|_\infty^p\mu(A) + \|u\|_2^2\right)^{\frac{1}{p}}$. (1.5 pts)
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(4) Consider the measure space $([0,\infty),\mathcal{B}([0,\infty)),\lambda)$, where $\mathcal{B}([0,\infty))$ is the Borel σ-algebra, and λ is Lebesgue measure on $[0,\infty)$. Let $f(x,y) = ye^{-(1+x^2)y^2}$ for $0 \le x, y < \infty$.

(a) Show that $f \in \mathcal{L}^1(\lambda\times\lambda)$, and determine the value of $\int_{[0,\infty)\times[0,\infty)} f\,d(\lambda\times\lambda)$. (1.5 pts)

(b) Prove that $\int_{[0,\infty)\times[0,\infty)} f\,d(\lambda\times\lambda) = \left(\int_{[0,\infty)} e^{-x^2}\,d\lambda(x)\right)^2$. Use part (a) to deduce the value of $\int_{[0,\infty)} e^{-x^2}\,d\lambda(x)$. (1.5 pts)