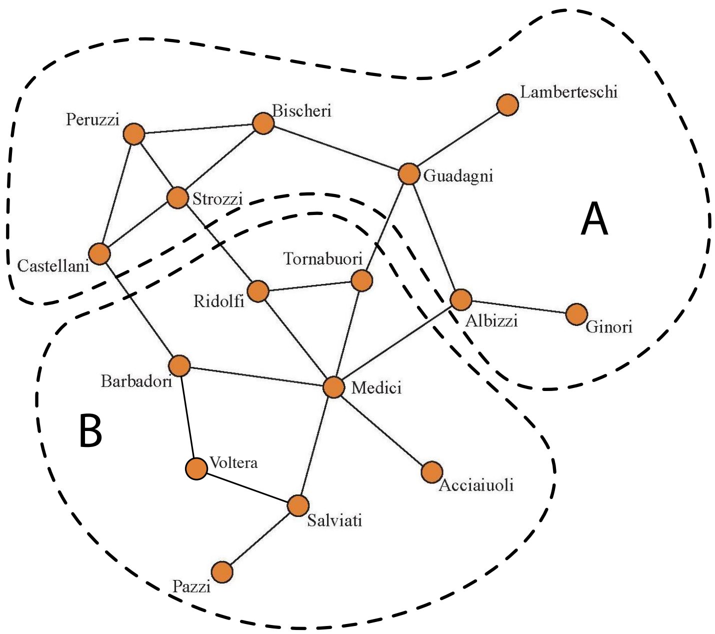
Solution model for: Methods and Models in Complex Systems BETA-B2CSA Final Examination

November 11, 2022

MAX 80 points, the score is capped at 70 points.



1. **Network analysis**

   The figure above, depicts the network of supposedly renaissance Florentine families, in which links represent business and marital ties. This network is impossible to cut into two connected components without removing as many as 4 edges, which, one may speculate, had something to do with relative stability of the network in times of a political turmoil. Nevertheless, being locked in a struggle for political control of the city of Florence in 1430s, two factions eventually appeared dominant in this struggle: one revolved around the powerful Strozzis (A), and the other around the infamous Medicis (B). One may reconstruct these factions, as shown, by computing the Fiedler's vector shifted to feature 0 median.

(a) [2 points] *Compute the cut quality Q of the indicated bisection.*

$$Q = \frac{4}{8 \cdot 8} = \frac{1}{16}.$$

(b) [3 points] *Show that the cut quality of the current partition cannot be improved by one family from* Medicis *(B) switching sides to* Strozzis *(A).* We use the first letter of the name that switches:

$$Q_T = Q_A = Q_R = Q_B = Q_P = \frac{5}{7 \cdot 9} = \frac{5}{63},$$

$$Q_V = \frac{6}{7 \cdot 9} = \frac{2}{21},$$

$$Q_M = \frac{9}{7 \cdot 9} = \frac{1}{7},$$

$$Q_S = \frac{7}{7 \cdot 9} = \frac{1}{9}.$$

(c) [2 points] *Calculate the empirical degree distribution p_k, that is the fraction of nodes with degree $k = 1, 2, 3, \ldots$*

$p_1 = \frac{1}{4}$, $p_2 = \frac{1}{16}$, $p_3 = \frac{1}{2}$, $p_4 = \frac{1}{8}$, $p_5 = 0$, $p_6 = \frac{1}{16}$, $p_k = 0$ for $k > 6$.

(d) [3 points] *We will now consider a modern network that has ties reaching out to a large number of nodes n. The structure of the network is not revealed, but the degree distribution p_k is known to be the same as in the Florentine network. Suppose one removes edges uniformly at random. Use the random graph formalism to answer what is the critical fraction of edges π_c that has to be kept* *to ensure that the network contains the giant component?*

$$\mu_1 = \sum_{k=1}^{6} kp_k = \frac{11}{4},$$

$$\mu_2 = \sum_{k=1}^{6} k^2 p_k = \frac{37}{4};$$

$$\pi_c = \frac{\mu_1}{\mu_2 - \mu_1} = \frac{11}{26} \approx 0.43...$$

2. **One-dimensional dynamics.** *Natural languages may become obsolete. A famous historical example is gradual abundance of Latin, even though speaking this language offered considerable socio-economical benefits. Language extinction is also active today, with 90 percent of world's languages are being expected to disappear by the end of this century.*

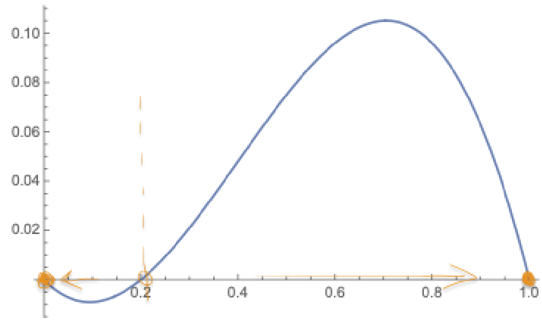
   Consider the following model of language competition: let X and Y denote two languages competing for speakers in a given society. The proportion of the population speaking X evolves according to

$$\dot{x} = (1-s)(1-x)x^2 - sx(1-x)^2$$

   where $0 \leq x \leq 1$ is the fraction of population speaking X and $1-x$ is the fraction of population speaking Y. Here $0 \leq s \leq 1$ is the socio-economical advantage of language Y over X.

   [10 points] Find the fixed points and classify their stability. Draw the 1D phase portrait. For what initial conditions a language offering a socio-economical advantage (e.g. for X, s being close to 0) may nevertheless become gradually abandoned.

   There are three fixed points:
   $x_1 = 0$ stable for $s > 0$ (unstable for $s = 0$)
   $x_2 = 1 - s$ unstable for $s \in (0, 1)$
   $x_3 = 1$ stable for $s < 1$ (unstable for $s = 1$)
   Furthermore $x_1 < x_2 < x_3$, for $s \in (0, 1)$.



If s is close to 0, but the initial fraction of population using it is small enough, i.e, $x_0 < 1 - s$, language X will be eventually be extinct.

3. **Dynamical system.** *With some infections, such as Covid'19, individuals may become temporary immune upon recovery. According to the current estimates, the immunity period is approximately 4 months, which is 8 times longer than the recovery period of two weeks. In the following model, we distinguish three compartments susceptible (S), infected (I), and immune/recovered (R). Consider the following modification of the SIR model:*

$$S + I \xrightarrow{\alpha} 2I$$

$$I \xrightarrow{\beta} R$$

$$R \xrightarrow{\beta/8} S$$

where $\alpha, \beta > 0$, $\alpha \neq \beta$ are the rates. Note that the rate of immunity loss is eight times smaller than the rate of recovery. Let $s(t)$, $x(t)$, and $r(t)$ denote concentration of correspondingly S, I and R species, with $s(t) + x(t) + r(t) = 1$

   (a) *[5 points] Formulate the system of ordinary differential equations for $s(t)$, $x(t)$ and $r(t)$ and show that this system can be well-represented by two differential equation for $s(t), x(t)$, write down these equations.*

$$\begin{cases} \frac{d}{dt}s(t) = \frac{\beta}{8}r(t) - \alpha s(t)x(t) \\ \frac{d}{dt}x(t) = -\beta x(t) + \alpha s(t)x(t) \\ \frac{d}{dt}r(t) = \beta x(t) - \frac{\beta}{8}r(t) \\ x(t)+s(t) + r(t) = 1, \end{cases}$$

which can be reduced by using $r(t) = 1 - s(t) - x(t)$:

$$\begin{cases} \frac{d}{dt}s(t) = \frac{\beta}{8}(1 - s(t) - x(t)) - \alpha s(t)x(t) \\ \frac{d}{dt}x(t) = -\beta x(t) + \alpha s(t)x(t) \end{cases}$$

(b) *[5 points] Write down the Jacobian matrix for this system of ODEs.*

$$\mathbf{J} = \begin{bmatrix} -\frac{\beta}{8} - \alpha x & -\frac{\beta}{8} - \alpha s \\ \alpha x & -\beta + \alpha s \end{bmatrix}$$

(c) *[10 points] Find all fixed points of the form (s^*, x^*), classify their stability depending on the parameters.*

FP1 (complete recovery):

$$s^* = 1, x^* = 0,\ r^* = 0$$

$$J_1 = \begin{bmatrix} -\frac{\beta}{8} & -(\alpha + \frac{\beta}{8}) \\ 0 & \alpha - \beta \end{bmatrix}$$

$$\tau = \alpha - \frac{9}{8}\beta,\ det = \frac{1}{8}(\beta - \alpha)\beta$$

FP2 (sustained positive fraction of infected and immune individuals):

$$s^* = \beta/\alpha,\ x^* = \frac{\alpha - \beta}{2\alpha},\ r^* = \frac{\alpha - \beta}{9\alpha},$$

$$\tau = -\frac{8\alpha + \beta}{72},\ det = \frac{1}{8}(\alpha - \beta)\beta$$

Hence: when $\alpha > \beta$ then FP1 is a saddle and FP2 is a stable node; when $\alpha < \beta$ then FP2 is a saddle and FP1 is a stable node.

4. **Markov chains.**

Consider a discrete dynamical system

$$x_{k+1} = Ax_k, \; k = 0, 1, 2 \ldots$$

where A is an $n \times n$ matrix with elements $a_{i,j} \in [0,1]$ and x_k are column vectors of length n.

(a) [10 points] *Let the columns of A sum up to 1, that is $\sum_{i=1}^{n} a_{i,j} = 1$. Show that if we start with an initial vector x_0 in which elements sum up to 1, then this normalisation property will be maintained for all x_k, $k > 0$.*

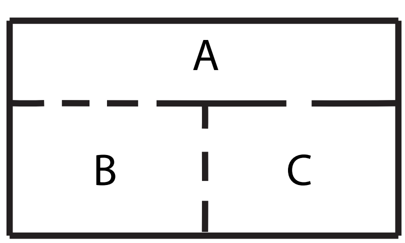
Let us expand the product $y = Ax$ and evaluate the sum of all element of y.

$$\sum_{i=1}^{n} (y)_i = \sum_{i=1}^{n} (Ax)_i = \sum_{i=1}^{n} \sum_{j=1}^{n} A_{i,j} x_j = \sum_{j=1}^{n} x_j \sum_{i=1}^{n} A_{i,j} =$$

$$= \sum_{j=1}^{n} x_j \cdot 1 = \sum_{j=1}^{n} x_j = 1.$$

where the last two equalities come from the fact that correspondingly the matrix columns sum up to 1 and vector x sums up to 1.

(b) [10 points] *Construct matrix A to represent the Markov chain for the following process. A trained mouse lives in the maze shown below. A bell rings at regular intervals, and the mouse is trained to change rooms (A,B, or C) each time it rings. When it changes rooms, it is equally likely to pass through any of the doors in the room it is in.*



We order the states in the alphabetic order (A,B,C), then:

$$\mathbf{M} = \begin{bmatrix} 0 & 3/5 & 1/3 \\ 3/4 & 0 & 2/3 \\ 1/4 & 2/5 & 0 \end{bmatrix}$$

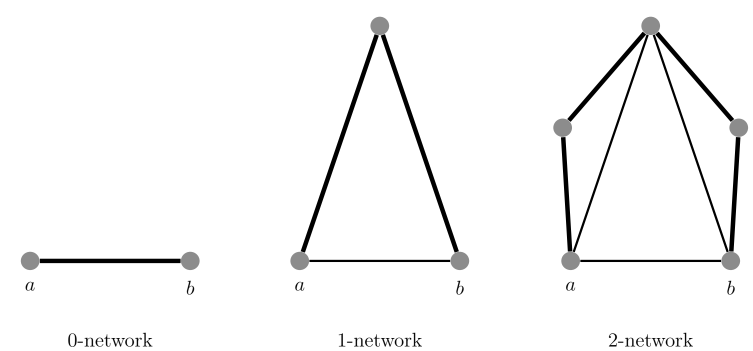
the column sum is indeed 1, therefore the matrix is left stochastic. Having this system in the matrix form we may, for example, compute, the fractions of mouse lifetime f_A,f_B,f_C spent in each room by solving $Mf = f$. Given that $f_C = 1 - f_A - f_B$, we have two equations:

$$3/5 f_B + 1/3(1 - f_A - f_B) = f_A$$

$$3/4 f_A + 2/3(1 - f_A - f_B) = f_B$$

leading to: $f_A = 1/3$, $f_B = 5/12$, $f_C = 1/4$.

5. **Hierarchical network**



Consider following iterative construction: A 0-network consist of two initial vertices a and b connected with a link. Iteratively, an $(n+1)$*-network is obtained from the n-network by glueing a triangle to each newly added link at the previous iteration, as shown.*

We remove each link with probability $1 - p$*. We say that a and b are connected with a path, if there is at least one way to travel from a to b by following the links. We are interested in* $f_n(p)$*, the probability that* *there is a path from a to b in a fringed n-network. Note that by definition, $f_0(p) = p$, because the only possible path is the link (a, b) itself.*

(a) Give a recursive equation for $f_n(p)$.

(b) Show that $f_n(p)$ converges for all $p \in (0, 1)$. For which values of p we have that $\lim_{n\to\infty} f_n(p) = 1$?

By definition, we have

$$f_0(p) = p.$$

In 1-network, there are two paths: one may take the initial link, which is present with probability $f_0(p) = p$, or, if it is not present, then there is still a detour of two links, which are simultaneously present with probability p^2. Hence,

$$f_1(p) = p + (1 - p)p^2,$$

and, in general, we have a recursion:

$$f_{n+1}(p) = A_p(f_n(p)) := p + (1 - p)f_n(p)^2.$$

One can see that $f_n(p)$ is a polynomial. This answers a).

To prove that $f_n(p)$ converges, first note that $f_1(p) > f_0(p)$. Assuming that $f_n(p) > f_{n-1}(p)$ we have

$$f_{n+1}(p) = p + (1 - p)f_n(p)^2 > p + (1 - p)f_{n-1}(p)^2 = f_n(p).$$

By induction, it follows that $f_n(p)$ is monotonically increasing sequence. Since also $f_n(p) \leq 1$ for all n, it follows that $f_n(p)$ converges.

We will now compute $F(p) := \lim_{n\to\infty} f_n(p)$. Since $A_p(x)$ is continuous, we know that $F(p)$ is a fixed point of A_p. Note that the equation

$$A_p(x) = x$$

has two roots, namely $x_1^* = 1$ and $x_2^* = \frac{p}{1-p}$.

To determine $F(p)$ we study the cases of $p > \frac{1}{2}$ and $p \leq \frac{1}{2}$ separately.

- Let $p > \frac{1}{2}$. Since $f_n(p) \leq 1$ we must have that $F(p) \leq 1$. Because, $p > \frac{1}{2}$, we have $x_2^* = \frac{p}{1-p} > 1$. Therefore, we find that $F(p) = x_1^* = 1$.

- Let $p \leq \frac{1}{2}$. In that case, we have $x_1^* \geq x_2^*$. Since $p < \frac{1}{2}$, we have $f_0(p) = p \leq \frac{p}{1-p}$. Assuming that $f_n(p) \leq \frac{p}{1-p}$, we find that

  $$f_{n+1}(p) = p + (1-p)f_n(p)^2 \leq p + (1-p)\frac{p^2}{(1-p)^2} = \frac{p}{1-p}.$$

  Using indunction, we conclude that $f_n(p) \leq \frac{p}{1-p} = x_2^*$ for all n. This implies that $F(p) \leq x_2^*$. Since $x_2^* \leq x_1^*$, it follows that $F(p) = x_2^* = \frac{p}{1-p}$.

Bringing these two cases together, we have

$$F(p) = \begin{cases} \frac{p}{1-p}, & p < \frac{1}{2}, \\ 1, & p \geq \frac{1}{2}. \end{cases}$$

Therefore $\lim\limits_{n\to\infty} f_n(p) = 1$ for $p \geq \frac{1}{2}$.