Methods and Models in Complex Systems
BETA-B2-CS2022 – Retake exam

January 4, 2023

([MAX 70 points] For this problems you will not need to use calculators or large matrix computations.)

1. Assume that we have a random graph, given by the configuration model with the degree distribution

$$p_k = a\delta_{k,1} + b\delta_{k,3} + c\delta_{k,9},\ a+b+c = 1$$

   Here $0 \le a, b, c \le 1$ are parameters of the model.

   (a) [10 pt] Determine the critical fraction f_c of vertices that we have to remove from the random graph with this degree distribution, to ensure that the giant component does not exist. Note that f_c depends on the parameters.

   (b) [10 pt] Let $c = 0$, compute the fraction of nodes in the largest connected component as a function of a and b.

2. The concentrations of S and I species in time are denoted by $s(t)$ and $x(t)$, satisfying $s(t) + x(t) = 1$. In the SIS model, the change of these quantities is governed by an infection process $S + I \to 2I$ having rate $\beta \ge 0$, and the recovery process $I \to S$ with rate $\delta \ge 0$.

   (a) [10 pt] Formulate the system of two ODEs for $s(t)$ and $x(t)$, find all fixed points of this system and study how their stability depends on the parameters δ and β. Explain the meaning of your findings in the context of epidemiology.

(b) [10 pt] Show that when the recovery rate $\delta = 0$ then SIS model can be reduced to the logistic equation.

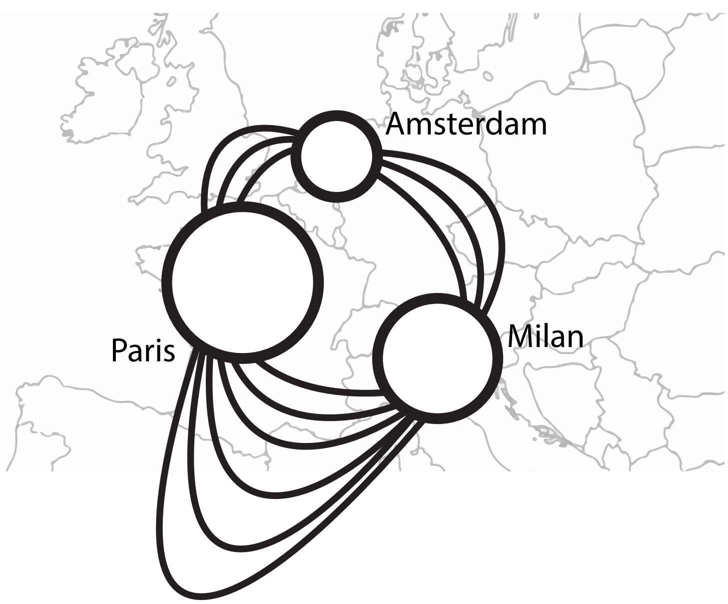
3. Consider a discrete map:

$$x_{n+1} = \alpha(x_n - x_n^2)$$

where $\alpha \geq 0$ is a parameter. It is known that as parameter α increases starting from 0, the map will experience a series of period doubling bifurcations and reach chaotic mode. Find values of parameter α for which this map has a period-two limit cycle. [10 points]

4. Three airports: Amsterdam, Milan and Paris are connected with multiple flights as shown (each flight connection is represented by one arc). You can see that the number of flight connections is different between different cities.

   Consider the following model for "lazy[1] random walks": every day a passenger decides to stay at the city with probability 1/2 or take a flight to another city with probability 1/2. The passenger is equally likely to take each of the outgoing flight conenctions.



[1]'lazy' refers to the fact that a passenger may decide to stay over and not travel on a given day.

(a) [10 pt] Write the transition probability matrix for passenger choices and use it to answer the following question: Suppose a passenger is in Paris, what is the probability the passenger will again be in Paris in two days (as a result of staying of making a round trip)?

(b) [10 pt] Given this is a representative model for all passengers and the model has reached its equilibrium, what fractions of passengers

$$0 \leq f_{\mathrm{Pa}}, f_{\mathrm{Mi}}, f_{\mathrm{Am}} \leq 1,\ f_{\mathrm{Pa}} + f_{\mathrm{Mi}} + f_{\mathrm{Am}} = 1$$

are at each city? Tip: note the symmetry between Paris and Milan.