Problem 1 Prove that if an action is strictly dominated, then it is never a best response, i.e., it is not a best response to any conjecture. [10 points]

A1 Let $(N, (A_i)_i, (u_i)_i)$ be a strategic-form game and suppose $a_i \in A_i$ is strictly dominated for player $i \in N$. So, there is $s_i \in \Delta(A_i)$ such that $u_i(s_i, a_{-i}) > u_i(a_i, a_{-i})$ for all $a_{-i} \in A_{-i}$. So, for any conjecture $\sigma_{-i} \in \Delta(A_{-i})$,

$$u_i(s_i, \sigma_{-i}) = \sum_{a_{-i}} \sigma_{-i}(a_{-i}) u_i(s_i, a_{-i}) > \sum_{a_{-i}} \sigma_{-i}(a_{-i}) u_i(a_i, a_{-i}) = u_i(a_i, \sigma_{-i}).$$

Hence, there is no conjecture $\sigma_{-i} \in \Delta(A_{-i})$ such that a_i is a best response to σ_{-i}. **NB:** Some students did not make the distinction between a profile $s_{-i} \in \prod_{j \neq i} \Delta(A_j)$ of mixed strategies and a conjecture $\sigma_{-i} \in \Delta(A_{-i})$; without this distinction, the proof is trivial.

Problem 2 There are two players. The players are roommates who each need to choose how much time to use to clean their apartment. That is, each player $i \in \{1, 2\}$ can choose an amount of time $t_i \geq 0$ to clean. If their choices are t_1, t_2, then player i's payoff is given by $u_i(t_i, t_{-i}) = (10 - t_{-i})t_i - t_i^2$. (Thus, the players dislike cleaning, and the more one roommate cleans, the less attractive for the other roommate it is to clean)

(a) What is the best response correspondence of each player i? [5 points]

(b) Which actions are rationalizable? [20 points]

(Note: You can restrict attention to pure strategies throughout.)

A2 Note that this game is strategically equivalent to the Cournot game in that it has the same best-response correspondence.

(a) Payoff functions are concave in a player's own strategy: For each player $i \in \{1, 2\}$, $d^2 u_i(t_i, t_{-i})/dt_i^2 < 0$. So, player i's best response $\mathrm{BR}_i(t_{-i})$ when the other player chooses $t_{-i} \geq 0$ is either 0 (boundary solution) or the point where $du_i(t_i, t_{-i})/dt_i = 0$, which is the case iff $t_i = \frac{1}{2}(10 - t_{-i})$.

(b) Let $i \in N$. Because $t_{-i} \geq 0 =: t^0$, choosing any action $t_i > 5$ is not a best response to any conjecture: Let $\varepsilon > 0$. Then, for any $t_{-i} \geq t^0$, $u_i(5, t_{-i}) - u_i(5 + \varepsilon, t_{-i}) = \varepsilon^2 + \varepsilon t_{-i} > 0$. So, if $t_i > 5$, then it is not a best response to any conjecture. It remains to show that any action $0 \leq t_i \leq 5 =: t^1$ is a best response to some conjecture. By monotonicity and continuity of the best response correspondence, it suffices to check that $t_i = 0$ and $t_i = 5$ are best responses to some conjecture. Now, $t_i = 0$ is a best response to the conjecture $t_{-i} = 10$, and $t_i = 5$ is a best response to the conjecture that $t_{-i} = 0$. Hence, $R_i^1 = [t^0, t^1]$, with $t^1 = \mathrm{BR}(t^0)$ for some (arbitrary) $i \in N$. By a similar argument, $R_i^1 = [t^2, t^1]$, where $t^2 := BR_i(t^1) = \frac{5}{2}$ for some $i \in N$.

For $k > 0$, suppose, inductively, that for $\ell \in \{1, \ldots, 2k\}$, we have $t^\ell = \text{BR}_i(t^{\ell-1})$ for some arbitrary $i \in N$ and that for each $i \in N$,

$$R_i^{2k-1} = [t^{2k-2}, t^{2k-1}];$$
$$R_i^{2k} = [t^{2k}, t^{2k-1}]$$

Let $i \in N$ and suppose i conjectures that the other player chooses an action $t_{-i} \geq t^{2k}$. Then, $BR_i(t_{-i}) \leq \frac{1}{2}(10 - t^{2k}) =: t^{2k+1}$, and by a similar argument as above, $R_i^{2k+1} = [t^{2k}, t^{2k+1}]$. Now suppose i conjectures that the other player chooses an action $t_{-i} \leq t^{2k+1}$. Then $BR_i(t_{-i}) \geq \frac{1}{2}(10 - t^{2k+1}) =: t^{2k+2}$, and by a similar argument as above, $R_i^{2k+2} = [t^{2k+2}, t^{2k+1}]$. The sequence $\{t^{2k}\}_{k\geq 0}$ is nondecreasing (because $R_i^{2k} \subseteq R_i^{2k-2}$ by definition) and bounded above (e.g., by t^1); hence, it converges to some t^*. Likewise, the sequence $\{t^{2k+1}\}_{k\geq 0}$ is nonincreasing (because $R_i^{2k+1} \subseteq R_i^{2k-1}$ by definition) and bounded (e.g., below by t^0); hence, it converges to some t^{**}. It follows that, for each $i \in N$, $R_i^\infty = [t^*, t^{**}]$, where $t^* = \text{BR}_i(t^{**})$ and $t^{**} = \text{BR}_i(t^*)$. Solving the system of equations $t^* = \text{BR}_i(t^{**})$ and $t^{**} = \text{BR}_i(t^*)$ gives $t^* = t^{**} = 10/3$. So, each player has a unique rationalizable strategy, and this is to choose $t_i = 10/3$. **NB1:** It is not strictly necessary to show (as was done above) that the strategies that lie strictly between the upper bound and lower bound (e.g., t^{2k+2}, t^{2k+1}) in a given stage are best responses to some conjecture (as long as one doesn't claim something like $R_i^{2k+2} = [t^{2k+2}, t^{2k+1}]$, as was done above); it suffices to note that the m-rationalizable strategies R_i^m at a given stage m must lie between the upper and lower bound at m (though there may be some strategy in this interval that are not in R_i^m) and then show that the upper and lower bounds converge to the same value (as was done above). **NB2:** It does *not*, in general, suffice to note that there exist unique t_1^*, t_2^* such that $t_i^* = BR_i(t_{-i}^*)$ for all $i \in \{1, 2\}$ (and to state that the unique rationalizable strategy for i is t_i^*). This is because the intersection of the best-response functions give the Nash equilibria, and in principle there can be rationalizable strategies that are not part of any Nash equilibria (see lecture notes L1, Example 2.15 or mock exam). Thus, if you want to find the rationalizable strategies from the intersection of the best-response functions then you need to argue why, in this case, this gives all the rationalizable strategies.

Problem 3 Consider the following game:

	ℓ	c	r
T	1,1	2,-2	-2,2
B	1,1	-2,2	2,-2

(a) Show that the game has no Nash equilibrium in which player 1 chooses a pure strategy. [5 points]

(b) Find all Nash equilibria of the game. [15 points]

A3 (a) Let s be a Nash equilibrium. If $s_1(T) = 1$, then the unique best response to s_1 is r, i.e., $s_2(r) = 1$; but the unique best response to r is B. If $s_1(B) = 1$, the unique best response to

s_1 is c; but the unique best response to c is T. So, $0 < s_1(T) < 1$. **NB:** Using the underlining method to show that the game does not have any pure Nash equilibria is not sufficient as this leaves open the possibility that player 2 mixes while player 1 plays a pure strategy.

(b) Let s be a Nash equilibrium and define $p := s_1(T)$. By (a), $0 < p < 1$. So, we must have $u_1(T, s_2) = u_1(B, s_2)$. If we define $q_1 := s_2(\ell)$, $q_2 := s_2(m)$, $q_3 := s_2(r)$, this gives $q_2 - 3q_3 = q_3 - 3q_2$ and thus $q_2 = q_3$. If $q_1 = 0$, we have $q_2 = q_3 = \frac{1}{2}$; but then the unique best response to s_2 is to play T; so by (a), this cannot be a Nash equilibrium. If $0 < q_1 < 1$, then $u_2(\ell, s_1) = u_2(c, s_1)$ and $u_2(\ell, s_1) = u_2(r, s_1)$. This gives $p = \frac{1}{4}$ and $p = \frac{3}{4}$, which obviously cannot hold simultaneously. So $q_1 = 1$. Then from $u_2(\ell, s_1) \geq u_2(c, s_1)$ and $u_2(\ell, s_1) \geq u_2(r, s_1)$, we get $\frac{1}{4} \leq p \leq \frac{3}{4}$. Hence, the set of Nash equilibria is

$$\{s \in S : s_2(\ell) = 1, \tfrac{1}{4} \leq s_1(T) \leq \tfrac{3}{4}\}.$$

NB1: It is *not* the case that ℓ is strictly dominated: For example, ℓ is a best response to the conjecture that player 1 chooses each of her two actions with equal probability. **NB2:** This is not a zero-sum game, so methods for solving zero-sum games (e.g., finding the lower envelope) do not apply here.

Problem 4 Find the value and optimal strategies of the following zero-sum game [20 points]:

$$\begin{pmatrix} 1 & -4 & 5 & 4 \\ 0 & 0 & 2 & 1 \\ 4 & 6 & 1 & 0 \end{pmatrix}$$

A4 Use IESDS: Action $j = 3$ is (strictly) dominated for player 2 by $j = 4$; once $j = 3$ has been eliminated, $i = 2$ is dominated for player 1 by the mixed strategy that puts probability $\frac{1}{2}$ on $i = 1$ and probability $\frac{1}{2}$ on $i = 3$; once $i = 2$ has been eliminated, $j = 1$ is strictly dominated for player 2 by the mixed strategy that puts probability $\frac{1}{2}$ on $j = 2$ and probability $\frac{1}{2}$ on $j = 4$. Once $j = 1$ has been eliminated, we have a (2×2) game that can be solved using equalizing strategies. We find that $x^T = (3/7, 0, 4/7)$ is optimal for player 1, $y^T = (0, 2/7/0, 5/7)$ is optimal for player 2, and the value is $12/7$. **NB:** If you do not consider (strict) dominance by mixed strategies, you can only eliminate $j = 3$ for player 2. That gives a (3×3) game with a nonsingular payoff matrix. However, it is not possible to use the methods from Section 1.1.1 in L4 to solve this (reduced) game, as the assumption that each player's optimal strategy has full support is incorrect.

Problem 5 Ann has two envelopes. She puts 10^n euro in one envelope, and 10^{n+1} euro in the other envelope, where the number n is chosen with equal probability from $\{1, 2, 3, 4, 5\}$. (For example, the probability that $n = 2$ is $\frac{1}{5}$.) She randomly hands one envelope to Bob and the other to Carol. (So, conditional on n, the probability that Bob has the envelope with 10^n euro is $\frac{1}{2}$, and likewise for Carol.) Bob and Carol are seated in different rooms and cannot communicate. Everyone knows how Ann has selected the amounts of money in each envelope and how the envelopes have been distributed among Bob and Carol.

(a) Write down the information structure, taking a state to be a pair (m, m') such that, if the state is (m, m'), the amount of money in Bob's envelope is 10^m while the amount of money in Carol's envelope is $10^{m'}$. [5 points]

(b) Show that for any state $\omega = (m, m')$, Bob's conditional expectation of the amount of money in Carol's envelope (given his information) strictly exceeds the amount of money in his own envelope if and only if $m \leq 5$. Likewise, for any state $\omega = (m, m')$, Carol's conditional expectation of the amount of money in Bob's envelope (given her information) strictly exceeds the amount of money in her own envelope if and only if $m' \leq 5$. [5 points]

Bob looks inside his envelope and finds that it contains 10^4 euros; Carol looks inside her envelope and finds that it contains 10^5 euros.

(c) Ann privately asks Bob and Carol whether they would be willing to switch envelopes; she then tells each of them what the other answered and repeats the question. Assuming that a player is willing to switch if and only if he/she expects the other's envelope to contain strictly more money than his/her own envelope, do players say "yes" or "no" when asked for the first time whether they are willing to switch? What about the second time? [15 points]

A5 (a) We refer to Bob as player 1 and to Carol as player 2, so $N = \{1, 2\}$. The information structure is $(\Omega, (\Pi_i)_{i\in N}, (\mathbb{P}_i)_{i\in N})$, with set of states

$$\Omega := \{(m, m') \in \mathbb{N}^2 : \exists n \in \{1, 2, 3, 4, 5, 6\} \text{ s.t. } m = n, m' = n + 1 \text{ or } m = n + 1, m' = n\}.$$

To define the information partitions, it will be convenient to define the functions $\beta : \Omega \to \{1, 2, 3, 4, 5, 6\}$ and $\gamma : \Omega \to \{1, 2, 3, 4, 5, 6\}$ by $\beta(m, m') = m$ and $\gamma(m, m') = m'$. Then:

$$\Pi_1 := \big\{\{\omega \in \Omega : \beta(\omega) = m\} : m \in \{1, 2, 3, 4, 5, 6\}\big\};$$
$$\Pi_2 := \big\{\{\omega \in \Omega : \gamma(\omega) = m'\} : m' \in \{1, 2, 3, 4, 5, 6\}\big\}.$$

Finally, the probability distributions for the players are equal to the uniform distribution over states, i.e., $\mathbb{P}_1 = \mathbb{P}_2 = \mathbb{P}$ where $\mathbb{P}$ is the probability distribution on Ω that assigns probability $1/10$ to each of the 10 states.

(b) We prove the result for Bob, the argument for Carol is analogous. Let $f_C : \Omega \to \{10^1, 10^2, \ldots, 10^6\}$ be the random variable that gives the amount of money in Carol's envelope. For $\omega \in \Omega$ such that $\beta(\omega) = 1$, $\mathbb{E}[f_C \mid \Pi_1(\omega)] = 10^2 > 10^1 = 10^{\beta(\omega)}$. For $\omega \in \Omega$ such that $\beta(\omega) \leq 5$, $\mathbb{E}[f_C \mid \Pi_1(\omega)] = \frac{1}{2}(10^{\beta(\omega)+1} + 10^{\beta(\omega)-1}) = 10^{\beta(\omega)}(5 + \frac{1}{20}) > 10^{\beta(\omega)}$. (Note that Bob knows $\beta(\omega)$ at ω, even if he does not know ω.)

(c) Denote the actual state by $\omega^* := (4, 5)$. By (b), both Bob and Carol answer "yes" at ω^* the first time the question is asked. Define

$$B_1^1 := \{\omega \in \Omega : \mathbb{E}[f_C \mid \Pi_1(\omega)] > \beta(\omega)\};$$
$$B_2^1 := \{\omega \in \Omega : \mathbb{E}[f_B \mid \Pi_2(\omega)] > \gamma(\omega)\},$$

where $f_B : \Omega \to \{10^1, 10^2, \ldots, 10^6\}$ is the random variable that gives the amount of money in Bob's envelope. So, $B_1^1 = \{\omega \in \Omega : \beta(\omega) \leq 5\}$ and $B_2^1 = \{\omega \in \Omega : \gamma(\omega) \leq 5\}$. After Ann has asked the question for the first time, the players' conditional expectations in ω^* are

$$\mathbb{E}[f_C \mid \Pi_1(\omega^*) \cap B_2^1] = \mathbb{E}[f_C \mid \Pi_1(\omega^*)] > \beta(\omega^*);$$
$$\mathbb{E}[f_B \mid \Pi_2(\omega^*) \cap B_1^1] = \beta(\omega^*) < \gamma(\omega^*).$$

So, the second time the question is asked, Bob says "yes," but Carol says "no."