Re-sit (hertentamen) Speltheorie (WISB272)
July 2023

- This is a **closed book exam**, and you are **not allowed to use a cheat sheet**.
- Put your ID card on the table. Turn off your cell phone and put it in your bag.
- You have 3 hours for the exam (plus an additional 30 minutes if you have extra time).
- The exam consists of five questions. The points you can earn for each (sub)question is indicated under each question. The total number of points you can earn is 100.
- Please **use a new sheet of paper for each question** and **write your name and student number on each sheet**. This will help avoid delays with grading.
- Your solutions can be in **English or in Dutch**. Try to be consistent in your choice of language so as to minimize the risk of confusion.
- Show your work on each problem. **If you are asked to prove a result, provide a formal proof**.
- **If you use a theorem or proposition from the lectures or lecture notes, clearly indicate this**. You do not need to name the result or provide its number; but don't forget to verify that the conditions of the statements you use have been met.
- **Good luck!**

Question 1 (new sheet of paper)

Let $G = (N, (A_i)_{i\in N}, (u_i)_{i\in N})$ be a finite strategic-form game. For $i \in N$, let $Z_i \subseteq A_i$ be a subset of actions. Suppose that for each player $i \in N$ and $z_i \in Z_i$, there is $z_{-i} \in Z_{-i}$ such that z_i is a best response to z_{-i}, i.e.,

$$\forall s_i \in \Delta(A_i): \qquad u_i(z_i, z_{-i}) \geq \sum_{a_i \in A_i} s_i(a_i) u_i(a_i, z_{-i}).$$

Prove, without using propositions or lemmas from the lecture notes, that every action $z_i \in Z_i$ is rationalizable. [10 points]

A1 Let $i \in N$ and $z_i \in Z_i$, and suppose that z_i is a best response to z_{-i} for some $z_{-i} \in Z_{-i}$. Then z_i is a best response to a conjecture σ_{-i} with support in R^0_{-i}, namely, the conjecture that the other players play according to z_{-i} (i.e., $\sigma_{-i}(z_{-i}) = 1$). Hence, $z_i \in R^1_i$. For $k > 1$, suppose that for $i \in N$ and $z_i \in Z_i$, we have that $z_i \in R^{k-1}_i$. Let $i \in N$ and $z_i \in Z_i$, and suppose that z_i is a best response to z_{-i} for some $z_{-i} \in Z_{-i}$. Then z_i is a best response to a conjecture σ_{-i} with support in R^{k-1}_{-i}, namely, the conjecture that the other players play according to z_{-i} (i.e., $\sigma_{-i}(z_{-i}) = 1$). Hence, $z_i \in R^k_i$. It now follows that for $i \in N$ and $z_i \in Z_i$, $z_i \in R^k_i$ for all $k \in \mathbb{N}$, that is, z_i is rationalizable. Note that one cannot use (without providing a proof) that every best-response set is a subset of the set of rationalizable action profiles.

Question 2 (new sheet of paper)

Consider the following game: Two players, Ann and Bob, pick a number from $\{0, 1, \ldots, 100\}$. If Ann picks a number $a \in \{0, 1, \ldots, 100\}$ and Bob picks a number $b \in \{0, 1, \ldots, 100\}$, her payoff is

$$u_{\text{Ann}}(a, b) = \begin{cases} 2 - (a-b)^2 & \text{if } a < b; \\ -(a-b)^2 & \text{otherwise.} \end{cases}$$

Similarly, if Ann picks a number $a \in \{0, 1, \ldots, 100\}$ and Bob picks a number $b \in \{0, 1, \ldots, 100\}$, Bob's payoff is

$$u_{\text{Bob}}(b, a) = \begin{cases} 2 - (a-b)^2 & \text{if } b < a; \\ -(a-b)^2 & \text{otherwise.} \end{cases}$$

(a) Show that for each player, any pure strategy (action) $c \in \{0, 1, \ldots, 99\}$ is a best response to some conjecture. [5 points]

(b) Determine the rationalizable strategies for both players. [20 points]

A2 (a) Each action $c < 100$ is a best response to the conjecture that the other player chooses $c + 1$.

(b) We use the equivalence between IESDS and rationalizability. For $k \in \{1, 2, \ldots\}$ and player $i \in N$, suppose $\Lambda_{-i}^{k-1} = \{0, \ldots, t^{k-1}\}$. (So, $t^0 = 100$.) By a similar argument as under (a), any action $c < t^{k-1}$ is a best response to the conjecture that the other player chooses $c + 1$. Furthermore, t^{k-1} is strictly dominated for i by $t^{k-1} - 1$: If the other player chooses $a_{-i} = t^{k-1}$, then $u_i(t^{k-1} - 1, t^{k-1}) = 1 > 0 = u_i(t^{k-1}, t^{k-1})$; if the other player chooses $a_{-i} < t^{k-1}$, then $u_i(t^{k-1} - 1, a_{-i}) = -(t^{k-1} - 1 - a_{-i})^2 > -(t^{k-1} - a_{-i})^2 = u_1(t^{k-1}, a_{-i})$. No other strategy can be eliminated at round k: any $a_i < t^{k-1}$ is a best response to the conjecture that the other player chooses $a_i + 1$. Hence, $\Lambda_i^k = \{0, \ldots, t^{k-1} - 1\}$. It now follows that the unique strategy that survives IESDS is to pick 0. Hence, the unique rationalizable strategy for each player is to pick 0.

Question 3 (new sheet of paper)

Consider the following game

	ℓ	c	r
T	5,5	3,0	0,2
M	5,1	2,1	1,0
B	0,0	2,5	4,2

(a) Find all rationalizable strategies. [5 points]

(b) Find all Nash equilibria. [15 points]

A3 (a) In a pure strategy is rationalizable iff it survives the iterated elimination of strictly dominated pure strategies. Strategy r is strictly dominated by the mixed strategy that puts equal probability on ℓ and c After r has been eliminated, strategy B is strictly dominated by T. After also B has been eliminated, no more pure strategies can be eliminated. Hence, the rationalizable pure strategies for player 1 are T and M, and for player 2 they are ℓ and c.

(b) It suffices to look among rationalizable strategies. Thus, it suffices to consider strategies for player 1 with support in T and M and to strategies for player 2 with support in ℓ and c. Let s_1 be the strategy for player 1 that puts probability $0 \leq p \leq 1$ on T and the remaining probability $1 - p$ on M; and let s_2 be the strategy for player 1 that puts probability $0 \leq q \leq 1$ on ℓ and the remaining probability $1 - q$ on c. Then T is a best response for player 1 against s_2 iff $0 \leq q \leq 1$, and M is a best response to s_2 iff $q = 1$. Likewise, ℓ is a best response against s_1 iff $0 \leq p \leq 1$, and c is a best response to s_1 iff $p = 0$. Hence, the set of Nash equilibria is

$$\{s \in S : s_1(B) = 0, s_2(r) = 0, s_2(\ell) = 1\}.$$

Question 4 (new sheet of paper)

Consider the zero-sum game with payoff matrix

$$\mathcal{A} = \begin{pmatrix} 2 & 0 & 1 & 4 \\ 1 & 2 & 5 & 3 \\ 4 & 1 & 3 & 2 \end{pmatrix}$$

Find the value and determine all optimal strategies for the players. [20 points]

A4 We can use IESDS to simplify the game, as the value and the set of optimal strategies remain unchanged when we iteratively eliminate strategies that are strictly dominated (lecture notes). Action $j = 2$ strictly dominates $j = 4$ for player 2. Once $j = 4$ has been eliminated, action $i = 3$ strictly dominates action $i = 1$ for player 1. Once $i = 1$ has been eliminated, $j = 2$ strictly dominates $j = 3$. This leaves us with the following reduced game:

$$\mathcal{B} = \begin{pmatrix} 1 & 2 \\ 4 & 1 \end{pmatrix}$$

This game has no saddle points. Using the method of equalizing payoffs (or drawing the best-response functions), we find that each player has a unique optimal strategy: $x^* = (0, 3/4, 1/4)^T$ is the unique optimal strategy for player 1, and $y^* = (1/4, 3/4, 0, 0)^T$ is the unique optimal strategy for player 2. The value is equal to $u_1(x^*, y^*)$ (lecture notes), so the value is $u_1(x^*, y^*) = u_1(1, y^*) = 7/4$.

Question 5 (new sheet of paper)

There are two players, labeled 1 and 2. The information structure is $(\Omega, (\Pi_i)_{i\in\{1,2\}}, (\mathbb{P}_i)_{i\in\{1,2\}})$, where $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6, \omega_7\}$, $\mathbb{P}_1 = \mathbb{P}_2 =: \mathbb{P}$ with

$$\mathbb{P}(\{\omega_1\}) = 4/32; \qquad \mathbb{P}(\{\omega_5\} = 7/32;$$
$$\mathbb{P}(\{\omega_2\}) = 2/32; \qquad \mathbb{P}(\{\omega_6\} = 2/32;$$
$$\mathbb{P}(\{\omega_3\}) = 8/32; \qquad \mathbb{P}(\{\omega_7\} = 4/32;$$
$$\mathbb{P}(\{\omega_4\}) = 5/32;$$

and

$$\Pi_1 = \big\{\{\omega_1, \omega_2, \omega_3\}, \{\omega_4, \omega_5, \omega_6\}, \{\omega_7\}\big\};$$
$$\Pi_2 = \big\{\{\omega_1, \omega_2, \omega_4\}, \{\omega_3, \omega_5, \omega_6, \omega_7\}\big\}.$$

Consider the event $E := \{\omega_1, \omega_3, \omega_4, \omega_5\}$, and suppose that the true state is ω_6.

(a) Calculate the (conditional) probability that each player assigns to E. [5 points]

(b) Assume that players repeatedly announce the conditional probability they assign to E, in the following way: In each round $k \in \{1, 3, 5, \dots\}$, player 1 announces the conditional probability that she assigns to E (given her information in round k); and in each round $k \in \{2, 4, 6, \dots\}$, player 2 announces the conditional probability he assigns to E (given his information in round k). Will players' conditional probabilities converge? If so, to what value do they converge, and in which round? Support your answer by specifying how the players' conditional probabilities change from one round to the next. [20 points]

A5 (a) Player 1 assigns conditional probability

$$\mathbb{P}(E \mid \Pi_1(\omega_6)) = \mathbb{P}(E \mid \{\omega_4, \omega_5, \omega_6\}) = (5+7)/(5+7+2) = 12/14$$

to E, and player 2 assigns conditional probability

$$\mathbb{P}(E \mid \Pi_2(\omega_6)) = \mathbb{P}(E \mid \{\omega_3, \omega_5, \omega_6, \omega_7\}) = (8+7)/(8+7+2+4) = 15/21$$

to E.

(b) In round 1, player 1 announces that she assigns probability 12/14 to E. Define

$$B_1^1 := \{\omega \in \Omega : \mathbb{P}(E \mid \Pi_1(\omega)) = 12/14\}$$

So, $B_1^1 = \{\omega_1, \omega_2, \omega_3, \omega_4, \omega_5, \omega_6\}$. In round 2 player 2 thus assigns probability $\mathbb{P}(E \mid \Pi_2(\omega_6) \cap B_1^1) = \mathbb{P}(E \mid \{\omega_3, \omega_5, \omega_6\}) = (8+7)/(8+7+2) = 15/17$ to E. Define

$$B_2^2 := \{\omega \in \Omega : \mathbb{P}(E \mid \Pi_2(\omega) \cap B_1^1) = 15/17\}.$$

So, $B_2^2 = \{\omega_3, \omega_5, \omega_6\}$. In round 3 player 1 thus assigns probability $\mathbb{P}(E \mid \Pi_1(\omega_6) \cap B_2^2) = \mathbb{P}(E \mid \{\omega_5, \omega_6\}) = 7/(7+2) = 7/9$ to E. Define

$$B_1^3 := \{\omega \in \Omega : \mathbb{P}(E \mid \Pi_1(\omega) \cap B_2^2) = 7/9\}.$$

So, $B_1^3 = \{\omega_5, \omega_6\}$. In round 4 player 2 thus assigns probability $\mathbb{P}(E \mid \Pi_2(\omega_6) \cap B_1^1 \cap B_1^3) = \mathbb{P}(E \mid \{\omega_3, \omega_5, \omega_6\} \cap \{\omega_5, \omega_6\}) = \mathbb{P}(E \mid \{\omega_5, \omega_6\}) = 7/9$ to E. Define

$$B_2^4 := \{\omega \in \Omega : \mathbb{P}(E \mid \Pi_2(\omega_6) \cap B_1^1 \cap B_1^3) = 7/9\};$$

then $B_2^4 = \{\omega_5, \omega_6\}$. Hence, in each round $k > 4$, each player assigns conditional probability 7/9 to E. The players' conditional probabilities thus converge; they converge in round 4, to 7/9.