Solutions for Exam Inleiding Topologie, February 2, 2023

Problem 1 (1+5 points)**.** *Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces.*

(i) For $A \subseteq X$, state the definition of the subspace topology on A.

(ii) Let $A, B \subseteq X$ be open subsets with $A \cup B = X$. Let $f\colon X \to Y$ be a map such that $f|_A\colon A \to Y$ and $f|_B\colon B \to Y$ are continuous, where we equip A and B with the subspace topology. Prove that f is continuous.

Solution: (i) The subspace topology is the collection $\mathcal{T}_X|_A$ of all subset $U \subset A$ such that there exists some $V \in \mathcal{T}_X$ with $V \cap A = U$.

(ii) Let A, B, f as above. We first observe that if $U \subset A$ is open in the subspace topology, then U is open in X as well. Indeed: $U = A \cap V$ for some open $V \subset X$. Since $A \subset X$ is open, its intersection with V is open in X as well. The analogous result holds for B.

Let now $U \subset Y$ be open. We have to show that $f^{-1}(U) \subset X$ is open. By definition of continuity, $(f|_A)^{-1}(U) \subset A$ and $(f|_B)^{-1}(U) \subset B$ are open in the subspace topology and hence also in X. Since $f^{-1}(U) = (f|_A)^{-1}(U) \cup (f|_B)^{-1}(U)$, we see that $f^{-1}(U)$ is open in X as well.

Problem 2 (3+4 points)**.** *Consider the set $\mathbb{N}$ of positive integers. Define $\mathcal{T}$ as the collection of all subsets of $\mathbb{N}$ of the form $U_n = \{k \in \mathbb{N} : k \geq n\}$ for $n \in \mathbb{N}$.*

(i) Show that $\mathcal{T}$ is not a topology, but one can make it into a topology by adding a single subset of $\mathbb{N}$.

(ii) Decide whether the resulting topological space is Hausdorff and whether it is second-countable (i.e. has a countable basis of topology).

Solution: (i) We observe:

- $U_i \cap U_j = U_{\max(i,j)} \in \mathcal{T}$,
- $\bigcup_{i \in I} U_i = U_{\min(i : i \in I)} \in \mathcal{T}$ for any set I of positive integers,
- $\mathbb{N} = U_1 \in \mathcal{T}$.

However, $\varnothing \notin \mathcal{T}$. Moreover, as $\varnothing \cap U_i = \varnothing$ and $\varnothing \cup U_i = U_i$, we see from the above that $\mathcal{T}' = \mathcal{T} \cup \{\varnothing\}$ is indeed a topology on $\mathbb{N}$.

(ii) The topological space $(\mathbb{N}, \mathcal{T}')$ is not Hausdorff. Indeed, U_1 is the only open neighborhood of 1 and $2 \in U_1$. Thus, we cannot find disjoint open neighborhoods of 1 and 2.

On the other hand, $(\mathbb{N}, \mathcal{T}')$ is second-countable: $\mathcal{T}'$ itself is a basis of topology and it has only countably many elements.

Problem 3 (1+9+5 points)**.** *Consider the subspaces*

- $A_1 = \{(x,y) \in \mathbb{R}^2 : y = 0\}$,
- $A_2 = \{(x,y) \in \mathbb{R}^2 : xy = 0\}$,
- $A_3 = \{(x,y) \in \mathbb{R}^2 : y = x^2\}$,
- $A_4 = \{(x,y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$.

of $\mathbb{R}^2$ *with the Euclidean topology.*

(i) Sketch $A_1, \dots, A_4$*.*

(ii) Which A_i *are manifolds?*

(iii) For every $1 \leq i < j \leq 4$ *decide whether* A_i *and* A_j *are homeomorphic.*

Solution: (i) Skipped. But in words: line, cross, parabola, circle.

(ii) We claim that A_1 and A_3 are homeomorphic to $\mathbb{R}$ with the Euclidean topology. Indeed, let $\mathrm{pr}_1 \colon \mathbb{R}^2 \to \mathbb{R}$ the projection onto the first coordinate. Then $\mathrm{pr}_1|_{A_1} \colon A_1 \to \mathbb{R}$ and $\mathrm{pr}_1|_{A_1} \colon A_3 \to \mathbb{R}$ are bijections and have the continuous inverses $x \mapsto (x, 0)$ and $x \mapsto (x, x^2)$, respectively. Since $\mathbb{R}$ is a manifold and being a manifold is a topological property, A_1 and A_3 are manifolds. Moreover, we have shown in class that $A_4 = S^1$ is a manifold. (Actually, all of these are one-dimensional.)

We claim that A_2 is not a manifold. Let U be an open neighborhood of 0 in A_2 with a homeomorphism $\varphi \colon \mathbb{R}^n \to U$. Thus, $U \setminus \{0\}$ is path-connected for $n \geq 2$, has exactly two path-components for $n = 1$ and is empty for $n = 0$ (since the corresponding statements are true for $\mathbb{R}^n \setminus \{0\}$). In summary: $U \setminus \{0\}$ has at most two path-components.

Consider now the sets $B_1 = \{(x,y) \in U : y > 0\}$, $B_2 = \{(x,y) \in U : y < 0\}$, $B_3 = \{(x,y) \in U : x > 0\}$ and $B_4 = \{(x,y) \in U : x < 0\}$. Since U is open (and has thus to contain the intersection of some open disk around 0 with A_3), all of these are non-empty. Moreover, there is no path connecting $p \in B_j$ and $q \in B_j$ for $j \neq k$ in $U \setminus \{0\}$. Indeed, for any such path $\gamma \colon [0,1] \to U \setminus \{0\}$, the preimages of the B_i would form a decomposition of $[0,1]$ into disjoint open subsets and at least $\gamma^{-1}(B_j)$ and $\gamma^{-1}(B_k)$ are non-empty; this is in contradiction to $[0,1]$ being connected. In summary: $U \setminus \{0\}$ must have at least four path-components, which is contradiction with our earlier conclusion that it has at most two path-components.

(iii) We have already seen in Part (ii) that A_1 and A_3 are homeomorphic (since they are both homeomorphic to $\mathbb{R}$). The space A_2 cannot be homeomorphic to A_1, A_3 or A_4 since the latter are manifolds, while A_2 isn't. Moreover, A_4 cannot be homeomorphic to A_1 or A_3 since A_4 is compact (as it is closed and bounded in $\mathbb{R}^2$), while A_1 and A_3 are not (they are not compact). [You could also just say that we know from class that S^1 and $\mathbb{R}$ are not homeomorphic.]

Problem 4 (6 points). *Equip the set $\mathcal{C}([0,1],\mathbb{R})$ of continuous functions from $[0,1]$ to $\mathbb{R}$ with the topology of uniform convergence, induced by the metric*

$$d(f,g) = \max_{x\in[0,1]} (|f(x) - g(x)|).$$

Show that there is no compact subset $K \subseteq \mathcal{C}([0,1],\mathbb{R})$ such that the functions

$$f_n\colon [0,1] \to \mathbb{R}, \qquad x \mapsto nx$$

are contained in K for all $n \in \mathbb{N}$.

Solution: Since every point has finite distance from the constant function 0, the balls $B_d(0,r)$ for r running over the positive real numbers form an open cover. If K is compact, it has thus to be contained in a finite number of $B_d(0,r)$ (choosing finite subcover) and is thus bounded. But the sequence of f_n is not bounded. [Alternatively, you can argue with Bolzano–Weierstraß.]

Problem 5 (4+4 points). *Let $\mathbb{P}^2 = D^2/\sim$ be the projective plane, with $\sim$ being the smallest equivalence relation such that $x \sim (-x)$ for all $x \in S^1 = \partial D^2$.*

(i) Give an example of an embedding $f\colon S^1 \to \mathbb{P}^2$ such that $\mathbb{P}^2 \setminus f(S^1)$ is path-connected.

(ii) Give examples of embeddings $f,g\colon S^1 \to \mathbb{P}^2$ such that $\mathbb{P}^2 \setminus f(S^1)$ is not homeomorphic to $\mathbb{P}^2 \setminus g(S^1)$.

Solution: (i) Let $p\colon D^2 \to \mathbb{P}^2$ be the quotient map. We learned in class that $S^1 \cong [0,1]/0 \sim 1$. Let $h\colon [0,1] \to D^2$ be the continuous function $t \mapsto (\cos(\pi t), \sin(\pi t))$. The composite ph sends 0 and 1 to the same point. Thus, the universal property of the quotient implies that we obtain a continuous function $\overline{h}\colon [0,1]/0 \sim 1$, sending $[t]$ to $p(\cos(\pi t), \sin(\pi t))$. This is injective and hence an embedding, since the source is compact and the target is Hausdorff (as a subspace of $\mathbb{P}^2$, which we showed in class to be Hausdorff). We obtain f by precomposing $\overline{h}$ with the homeomorphism $S^1 \to [0,1]/0 \sim 1$. We observe that $\mathbb{P}^2 \setminus f(S^1) = p(D^2 \setminus S^1)$ and thus path-connected as the image of a path-connected space under a continuous map.

(ii) We use f as in Part (i). It suffices to find an embedding $g\colon S^1 \to \mathbb{P}^2$ such that $\mathbb{P}^2 \setminus g(S^1)$ is not path-connected. We define $g(x)$ as $p(\frac{1}{2}x)$ (clearly injective and hence embedding as above). By the definition of the quotient topology,

$$p(\{(x,y) : ||(x,y)|| > \frac{1}{2}) \quad \text{and} \quad p(\{(x,y) : ||(x,y)|| < \frac{1}{2})$$

are open in $\mathbb{P}^2$ and hence in $\mathbb{P}^2 \setminus g(S^1)$. These form a decomposition of $\mathbb{P}^2 \setminus g(S^1)$ into two disjoint non-empty open subsets. Thus, $\mathbb{P}^2 \setminus g(S^1)$ is not connected and hence not path-connected.

Problem 6 (8 points)**.** *Let M be the Möbius strip, i.e. the quotient of $[0,1]^2$ by the smallest equivalence relation $\sim$ on $[0,1]^2$ such that $(0,t) \sim (1,1-t)$. Define an action of the group $(\mathbb{Z},+)$ on $\mathbb{R} \times [0,1]$ such that the quotient of $\mathbb{R} \times [0,1]$ by this group action is homeorphic to M. (Here, we equip $\mathbb{R} \times [0,1] \subset \mathbb{R}^2$ with the subspace topology of the Euclidean topology.)*

Solution: Let $f\colon \mathbb{R} \times [0,1] \to \mathbb{R} \times [0,1]$ be the map $(x,t) \mapsto (x+1, 1-t)$, which is a homeomorphism. Define the action of $(\mathbb{Z},+)$ on $\mathbb{R} \times [0,1]$ by $n.(x,t) = f^n(x,t)$; here $f^0 = \mathrm{id}$, f^n for positive n is the n-fold iterated application of the map f and f^n for negative n is $(f^{-1})^n$. One easily checks that this is a group action.

Let $p\colon \mathbb{R} \times [0,1] \to X = (\mathbb{R} \times [0,1])/\mathbb{Z}$ be the quotient map. Composing the inclusion $i\colon [0,1] \times [0,1] \to \mathbb{R} \times [0,1]$ with p defines a map $pi\colon [0,1]^2 \to X$. By the universal property of the quotient, this factors through a continuous map $h\colon M \to X$ (indeed: $(1, 1-t) = f(0,t)$). The map h is surjective, as we can write every x as $n+s$ for $s \in [0,1)$ and thus $(x,t) = f^n(s,t)$ if n is even or $(s, 1-t)$ if n is odd and thus $p(x,t) = p(s,t)$ or $p(s,1-t)$. Moreover, it is easily seen to be injective. Thus, h is a continuous bijection. Since M is compact (as we know from class), it suffices to show that X is Hausdorff to conclude that h is a homeomorphism.

Let $p(x,t)$ and $p(y,s)$ be two different points in X. Let

$$r = \min_{n \in \mathbb{Z}} (d((x,t), f^n(y,s))).$$

Then $p(B_d((x,t), \frac{r}{2}))$ and $p(B_d((x,t), \frac{r}{2}))$ are disjoint. Moreover, they are open as the quotient map $\mathbb{R} \times [0,1] \to X$ sends open maps to open sets (as every quotient by a *group action* does, as shown in class). Thus, X is Hausdorff.